Chapter 1

Fundamentals of Engineering Economy

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General Objective:

Trainee will be able to understand the basic concepts and terminology necessary for engineering economy

Detailed Objectives:

1. Engineering economics: description and role in decision making process.
2. Performing an Engineering Economy Study.
3. Interest Rate and Rate of Return (ROR).
4. Economic Equivalence.
5. Terminology and Symbols.
6. Simple and Compound Interest.
7. Cash Flows: Their Estimation and Diagramming (CFD).
8. Minimum Attractive Rate of Return (MARR).
9. Spreadsheets use in engineering economy.

Introduction

The need for engineering economy is primarily motivated by the work that engineers do in performing analyses, synthesizing, and coming to a conclusion as they work on projects of all sizes. In other words, engineering economy is at the heart of ***making decisions.*** These decisions involve the fundamental elements of ***cash flows of money, time, and interest rates.*** This chapter introduces the basic concepts and terminology necessary for an engineer to combine these three essential elements in organized,

1. Engineering economics: description and role in decision making process

Decisions are made routinely to choose one alternative over another by individuals in everyday life; by engineers on the job; by managers who supervise the activities of others; by corporate presidents who operate a business; and by government officials who work for the public good. Most decisions involve money, called capital or capital funds, which is usually limited in amount. The decision of where and how to invest this limited capital is motivated by a primary goal of adding value as future, anticipated results of the selected alternative are realized. Engineers play a vital role in capital investment decisions based upon their ability and experience to design, analyze, and synthesize. The factors upon which a decision is based are commonly a combination of economic and noneconomic elements. Engineering economy deals with the economic factors. By definition,

> ***Engineering economy involves formulating, estimating, and evaluating the expected economic outcomes of alternatives designed to accomplish a defined purpose. Mathematical techniques simplify the economic evaluation of alternatives.***

Other terms that mean the same as engineering economy are engineering economic analysis, capital allocation study, economic analysis, and similar descriptors.

People make decisions; computers, mathematics, concepts, and guidelines assist people in their decision-making process. Since most decisions affect what will be done, the time frame of engineering economy is primarily the future. Therefore, the numbers used in engineering economy are ***best estimates of what is expected to occur***. The estimates and the decision usually involve four essential elements:

- Cash flows
- Times of occurrence of cash flows
- Interest rates for time value of money
- Measure of economic worth for selecting an alternative

Since the estimates of cash flow amounts and timing are about the future, they will be somewhat different than what is actually observed, due to changing circumstances and unplanned events. In short, the variation between an amount or time estimated now and that observed in the future is caused by the stochastic (random) nature of all economic events. ***Sensitivity analysis*** is utilized to determine how a decision might change according to varying estimates, especially those expected to vary widely. Example (1-1) illustrates the fundamental nature of variation in estimates and how this variation may be included in the analysis at a very basic level.

Example (1-1)

An engineer is performing an analysis of warranty costs for drive train repairs within the first year of ownership of luxury cars purchased in the United States. He found the average cost (to the nearest dollar) to be $570 per repair from data taken over a 5-year period.

Year	2006	2007	2008	2009	2010
Average Cost, $/repair	525	430	619	650	625

What range of repair costs should the engineer use to ensure that the analysis is sensitive to changing warranty costs?

Solution

At first glance the range should be approximately –25% to +15% of the $570 average cost to include the low of $430 and high of $650. However, the last 3 years of costs are higher and more consistent with an average of $631. The observed values are approximately ±3% of this more recent average.

- If the analysis is to use the most recent data and trends, a range of, say, ±5% of $630 is recommended.
- If, however, the analysis is to be more inclusive of historical data and trends, a range of, say, ±20% or ±25% of $570 is recommended.

The criterion used to select an alternative in engineering economy for a specific set of estimates is called a measure of worth. The measures developed and used in this text are:

- Present worth (PW)
- Future worth (FW)
- Annual worth (AW)
- Rate of return (ROR)
- Benefit/cost (B/C)
- Capitalized cost (CC)
- Payback period (PP)
- Economic value added (EVA)
- Cost Effectiveness (CE)

All these measures of worth account for the fact that money makes money over time. This is the concept of the time ***value of money***.

It is a well-known fact that money makes money. The time value of money explains the change in the amount of money over time for funds that are owned (invested) or owed (borrowed). This is the most important concept in engineering economy.

The time value of money is very obvious in the world of economics. If we decide to invest capital (money) in a project today, we inherently expect to have more money in the future than we invested. If we borrow money today, in one form or another, we expect to return the original amount plus some additional amount of money. Engineering economics is equally well suited for the future and for the analysis of past cash flows in order to determine if a specific criterion (measure of worth) was attained.

2. Performing an Engineering Economy Study

An engineering economy study involves many elements: problem identification, definition of the objective, cash flow estimation, financial analysis, and decision making. Implementing a structured procedure is the best approach to select the best solution to the problem.

The steps in an engineering economy study are as follows:

1- Identify and understand the problem; identify the objective of the project.
2- Collect relevant, available data and define viable solution alternatives.
3- Make realistic cash flow estimates.
4- Identify an economic measure of worth criterion for decision making.

5- Evaluate each alternative; consider noneconomic factors; use sensitivity analysis as needed.
6- Select the best alternative.
7- Implement the solution and monitor the results.

Technically, the last step is not part of the economy study, but it is, of course, a step needed to meet the project objective. There may be occasions when the best economic alternative

requires more capital funds than are available, or significant noneconomic factors preclude the most economic alternative from being chosen. Accordingly, steps 5 and 6 may result in selection of an alternative different from the economically best one. Also, sometimes more than one project may be selected and implemented. This occurs when projects are independent of one another. In this case, steps 5 through 7 vary from those above. Figure (1-1) illustrates the steps above for one alternative. Descriptions of several of the elements in the steps are important to understand.

Problem Description and Objective Statement: A succinct statement of the problem and primary objective(s) is very important to the formation of an alternative solution. As an illustration, assume the problem is that a coal-fueled power plant must be shut down by 2030 due to the production of excessive sulfur dioxide. The objectives may be to generate the forecasted electricity needed for 2030 and beyond, plus to not exceed all the projected emission allowances in these future years.

Alternatives: These are stand-alone descriptions of viable solutions to problems that can meet the objectives. Words, pictures, graphs, equipment and service descriptions, simulations, etc. define each alternative. The best estimates for parameters are also part of the alternative. Some parameters include equipment first cost, expected life, salvage value (estimated trade-in, resale, or market value), and annual operating cost (AOC), which can also be termed maintenance and operating (M&O) cost, and subcontract cost for specific services. If changes in income (revenue) may occur, this parameter must be estimated.

Cash Flows: All cash flows are estimated for each alternative. Since these are future expenditures and revenues, the results of step 3 usually prove to be inaccurate when an alternative is actually in place and operating. When cash flow estimates for specific parameters are expected to vary significantly from a point estimate made now, risk and sensitivity analyses (step 5) are needed to improve the chances of selecting the best alternative. Sizable variation is usually expected in estimates of revenues, AOC, salvage values, and subcontractor costs.

Engineering Economy Analysis: The techniques and computations that you will learn and use throughout this text utilize the cash flow estimates, time value of money, and a selected measure of worth. The result of the analysis will be one or more numerical values; this can be in one of several terms, such as money, an interest rate, number of years, or a probability. In the end, a selected measure of worth mentioned in the previous section will be used to select the best alternative.

Before an economic analysis technique is applied to the cash flows, some decisions about what to include in the analysis must be made. Two important possibilities are taxes and inflation. Federal, state or provincial, county, and city taxes will impact the costs of every alternative. An after-tax analysis includes some additional estimates and methods compared to a before-tax analysis. If taxes and inflation are expected to impact all alternatives equally, they may be disregarded in the analysis. However, if the size of these projected costs is important, taxes and inflation should be considered. Also, if the impact of inflation over time is important to the decision, an additional

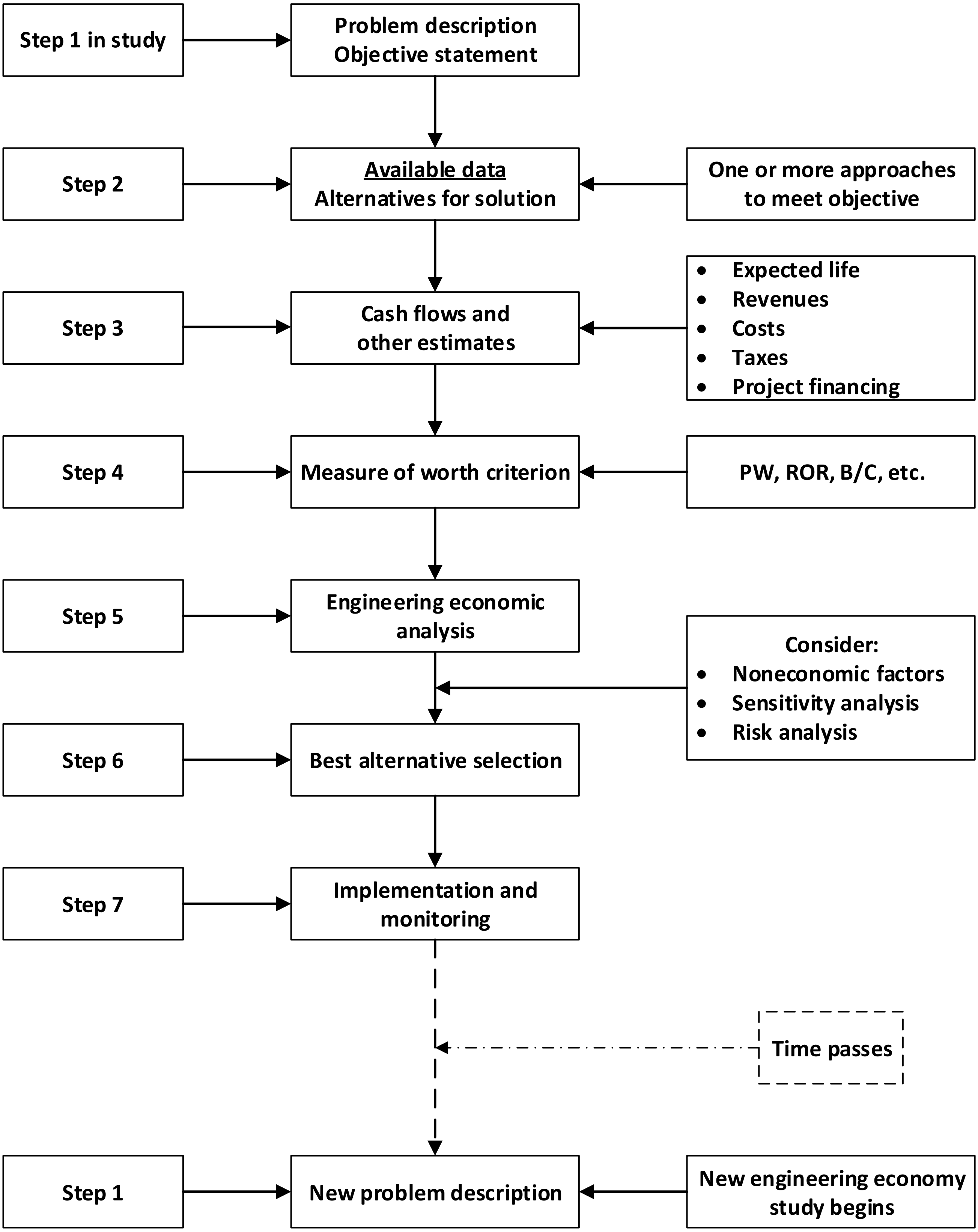


Figure (1-1): Steps in an engineering economy study

Selection of the Best Alternative: The measure of worth is a primary basis for selecting the best economic alternative. For example, if alternative A has a rate of return (ROR) of 15.2% per year and alternative B will result in an ROR of 16.9% per year, B is better economically. However, there can always be ***noneconomic*** or ***intangible factors*** that must be considered and that may alter the decision. There are many possible noneconomic factors; some typical ones are:

- Market pressures, such as need for an increased international presence
- Availability of certain resources, e.g., skilled labor force, water, power, tax incentives
- Government laws that dictate safety, environmental, legal, or other aspects
- Corporate management's or the board of director's interest in a particular alternative
- Goodwill offered by an alternative toward a group: employees, union, county, etc.

As indicated in Figure (1-1), once all the economic, noneconomic, and risk factors have been evaluated, a final decision of the "best" alternative is made. At times, only one viable alternative is identified. In this case, the do-nothing (DN) alternative may be chosen provided the measure of worth and other factors result in the alternative being a poor choice. The do-nothing alternative maintains the status quo.

3. Interest Rate and Rate of Return (ROR)

Interest is the manifestation of the time value of money. Computationally, interest is the difference between an ending amount of money and the beginning amount. If the difference is zero or negative, there is no interest. There are always two perspectives to an amount of interest—interest ***paid*** and interest earned. These are illustrated in Figure (1–2). Interest is paid when a person or organization borrowed money (obtained a loan) and repays a larger amount over time. Interest is ***earned*** when a person or organization saved, invested, or lent money and obtains a return of a larger amount over time. The numerical values and formulas used are the same for both perspectives, but the interpretations are different.

Interest paid on borrowed funds (a loan) is determined using the original amount, also called the principal,

$$\boldsymbol{Interest} = \boldsymbol{amount\ owed\ now} - \boldsymbol{principal} \qquad (1-1)$$

When interest paid over a specific time unit is expressed as a percentage of the principal, the result is called the ***interest rate***.

$$\boldsymbol{Interest\ rate}\ (\%) = \frac{\boldsymbol{interest\ accrued\ per\ time\ unit}}{\boldsymbol{principal}} \times \mathbf{100}\% \qquad (1-2)$$

The time unit of the rate is called the ***interest period***. By far the most common interest period used to state an interest rate is 1 year. Shorter time periods can be used, such as 1% per month. Thus, the interest period of the interest rate should always be included. If only the rate is stated, for example, 8.5%, a 1-year interest period is assumed.

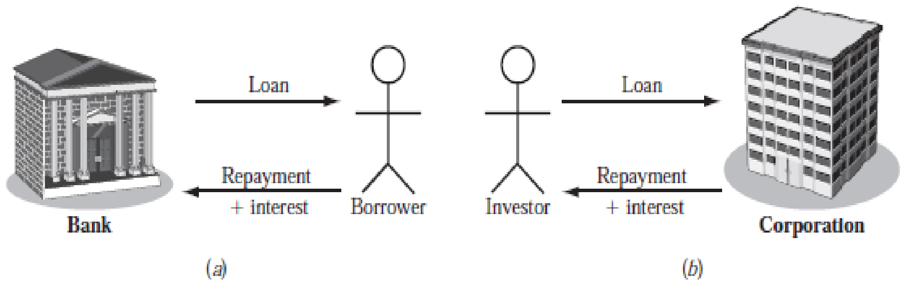


Figure (1-2): (a) Interest paid over time to lender. (b) Interest earned over time by investor.

Example (1-2)

An employee at LaserKinetics.com borrows \$10,000 on May 1 and must repay a total of \$10,700 exactly 1 year later. Determine the interest amount and the interest rate paid.

Solution

The perspective here is that of the borrower since \$10,700 repays a loan. Apply Equation (1-1) to determine the interest paid.

$$Interest\ paid = \$10{,}700 - 10{,}000 = \$700$$

Equation (1-2) determines the interest rate paid for 1 year.

$$Percent\ interest\ rate\ (\%) = \frac{\$700}{\$10{,}000} \times 100\% = 7\%\ per\ year$$

Example (1-3)

Stereophonics, Inc., plans to borrow \$20,000 from a bank for 1 year at 9% interest for new recording equipment.

1- Compute the interest and the total amount due after 1 year.
2- Construct a column graph that shows the original loan amount and total amount due after 1 year used to compute the loan interest rate of 9% per year.

Solution

1- Compute the total interest accrued by solving Equation (1-2) for interest accrued.

$$Interest = \$20{,}000 \times 0.09 = \$1800$$

The total amount due is the sum of principal and interest.

$$Total\ due = \$20{,}000 + 1800 = \$21{,}800$$

2- Figure (1-3) shows the values used in Equation (1-2): \$1800 interest, \$20,000 original loan principal, 1-year interest period.

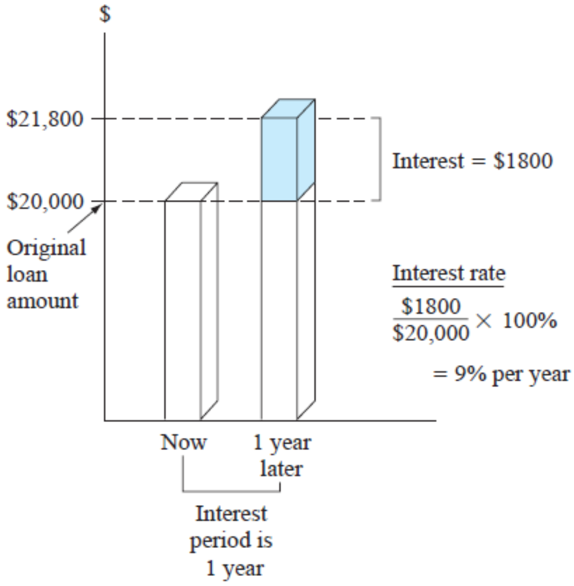


Figure (1-3): Values used to compute an interest rate of 9% per year.

Example (1-3)

From the perspective of a saver, a lender, or an investor, ***interest earned*** (Figure (1–2b)) is the final amount minus the initial amount, or principal.

$$\boldsymbol{interest\ earned\ =\ total\ amount\ now -\ principal} \qquad (1-3)$$

Interest earned over a specific period of time is expressed as a percentage of the original amount and is called ***rate of return (ROR).***

$$\boldsymbol{Rate\ of\ return}\ (\%) = \frac{\boldsymbol{interest\ accrued\ per\ time\ unit}}{\boldsymbol{principal}} \times \mathbf{100}\% \quad (1-4)$$

The time unit for rate of return is called the interest period, just as for the borrower's perspective. Again, the most common period is 1 year. The term return on investment (ROI) is used equivalently with ROR in different industries and settings, especially where large capital funds are committed to engineering-oriented programs. The numerical values in Equations (1-2) and (1-4) are the same, but the term interest rate paid is more appropriate for the borrower's perspective, while the rate of return earned is better for the investor's perspective.

Example (1-4)

1- Calculate the amount deposited 1 year ago to have $1000 now at an interest rate of 5% per year.
2- Calculate the amount of interest earned during this time period.

Solution

1- The total amount accrued ($1000) is the sum of the original deposit and the earned interest. If X is the original deposit,

$$Total\ accrued = deposit + deposit(interest\ rate)$$
$$\$1000 = X + X(0.05) = X(1 + 0.005) = 1.05X$$

The original deposit is

$$X = \frac{1000}{1.05} = \$952.38$$

2- Apply Equation (1-3) to determine the interest earned.

$$Interest = \$1000 - 952.38 = \$47.62$$

In Examples (1-2) to (1-4) the interest period was 1 year, and the interest amount was calculated at the end of one period. When more than one interest period is involved, e.g., the amount of interest after 3 years, it is necessary to state whether the interest is accrued on a simple or compound basis from one period to the next. This topic is covered later in this chapter.

Since ***inflation*** can significantly increase an interest rate, some comments about the fundamentals of inflation are warranted at this early stage. Inflation represents a decrease in the value of a given currency. That is, $10 now will not purchase the same amount of gasoline for your car (or most other things) as $10 did 10 years ago. The changing value of the currency affects market interest rates.

4. Terminology and Symbols

The equations and procedures of engineering economy utilize the following terms and symbols. Sample units are indicated.

symbols	***Definitions***
P	Value or amount of money at a time designated as the present or time 0. Also, P is referred to as present worth (PW), present value (PV), net present value (NPV), discounted cash flow (DCF), and capitalized cost (CC); monetary units, such as dollars
F	Value or amount of money at some future time. Also, F is called future worth (FW) and future value (FV); dollars
A	Series of consecutive, equal, end-of-period amounts of money. Also, A is called the annual worth (AW) and equivalent uniform annual worth (EUAW); dollars per year, euros per month
n	Number of interest periods; years, months, day
i	Interest rate per time period; percent per year, percent per month
t	Time, stated in periods; years, months, days

Example (1-5)

Today, Julie borrowed $5000 to purchase furniture for her new house. She can repay the loan in either of the two ways described below. Determine the engineering economy symbols and their value for each option.

1- Five equal annual installments with interest based on 5% per year.
2- One payment 3 years from now with interest based on 7% per year.

Solution

1- The repayment schedule requires an equivalent annual amount A, which is unknown.

P = \$5000 i = 5% per year n = 5 years A = ?

2- Repayment requires a single future amount F, which is unknown.

P = \$5000 i = 7% per year n = 3 years F = ?

Example (1-6)

You plan to make a lump-sum deposit of \$5000 now into an investment account that pays 6% per year, and you plan to withdraw an equal end-of-year amount of \$1000 for 5 years, starting next year. At the end of the sixth year, you plan to close your account by withdrawing the remaining money. Define the engineering economy symbols involved.

Solution

All five symbols are present, but the future value in year 6 is the unknown.

P = \$5000
A = \$1000 per year for 5 years
F = ? at end of year 6
i = 6% per year
n = 5 years for the A series and 6 for the F value

Example (1-7)

Last year Jane's grandmother offered to put enough money into a savings account to generate \$5000 in interest this year to help pay Jane's expenses at college.

1- Identify the symbols,
2- Calculate the amount that had to be deposited exactly 1 year ago to earn \$5000 in interest now, if the rate of return is 6% per year.

Solution

1- Symbols P (last year is -1) and F (this year) are needed.

P = ?
i = 6% per year
n = 1 year
$F = P$ + interest = ? + \$5000

2- Let F = total amount now and P = original amount. We know that $F – P = \$5000$ is accrued interest. Now we can determine P. Refer to Equations (1-1) through (1- 4).

$$F = P + Pi$$

The \$5000 interest can be expressed as

$$Interest = F - P = P + Pi - P = Pi$$

$$\$5000 = P(0.06); \quad P = \frac{\$5000}{0.06} = \$83{,}333.33$$

5. Cash Flows: Estimation and Diagramming (CFD)

As mentioned in earlier sections, cash flows are the amounts of money estimated for future projects or observed for project events that have taken place. All cash flows occur during specific time periods, such as 1 month, every 6 months, or 1 year. Annual is the most common time period. For example, a payment of \$10,000 once every year in December for 5 years is a series of 5 outgoing cash flows. And an estimated receipt of \$500 every month for 2 years is a series of 24 incoming cash flows. Engineering economy bases its computations on the timing, size, and direction of cash flows.

Cash inflows are the receipts, revenues, incomes, and savings generated by project and business activity. A plus sign indicates a cash inflow.

Cash outflows are costs, disbursements, expenses, and taxes caused by projects and business activity. A negative or minus sign indicates a cash outflow. When a project involves only costs, the minus sign may be omitted for some techniques, such as benefit/cost analysis.

Of all the steps in Figure (1–1) that outline the engineering economy study, estimating cash flows (step 3) is the most difficult, primarily because it is an attempt to predict the future. Some examples of cash flow estimates are shown here. As you scan these, consider how the cash inflow or outflow may be estimated most accurately.

Cash Inflow Estimates:

Income: -$150,000 per year from sales of solar-powered watches
Savings: -$24,500 tax savings from capital loss on equipment salvage
Receipt: -$750,000 received on large business loan plus accrued interest
Savings: -$150,000 per year saved by installing more efficient air conditioning
Revenue: -$50,000 to -$75,000 per month in sales for extended battery life iPhones

Cash Outflow Estimates:

Operating costs: -$230,000 per year annual operating costs for software services
First cost: -$800,000 next year to purchase replacement earthmoving equipment
Expense: -$20,000 per year for loan interest payment to bank
Initial cost: -$1 to -$1.2 million in capital expenditures for a water recycling unit

All of these are ***point estimates,*** that is, ***single-value estimates*** for cash flow elements of an alternative, except for the last revenue and cost estimates listed above. They provide a ***range estimate,*** because the persons estimating the revenue and cost do not have enough knowledge or experience with the systems to be more accurate. Once all cash inflows and outflows are estimated (or determined for a completed project), the ***net cash flow*** for each time period is calculated.

$$\boldsymbol{Net\ cash\ flow\ = cash\ inflows - \ cash\ outflows} \qquad (1-5)$$

$$\boldsymbol{NCF\ = R - \ D} \qquad (1-6)$$

where NCF is net cash flow, R is receipts, and D is disbursements.

At the beginning of this section, the ***timing, size, and direction of cash flows*** were mentioned as important. Because cash flows may take place at any time during an interest period, as a matter of convention, all cash flows are assumed to occur at the end of an interest period.

The end-of-period convention means that all cash inflows and all cash outflows are assumed to take place at the end of the interest period in which they actually occur. When several inflows and outflows occur within the same period, the net cash flow is assumed to occur at the end of the period.

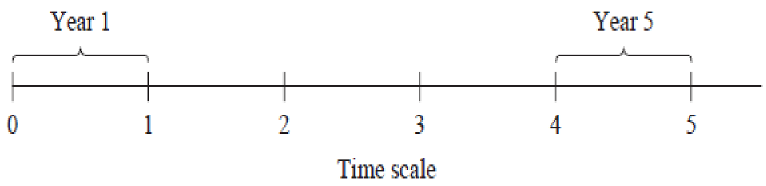


Figure (1-4): A typical cash flow time scale for 5 years.

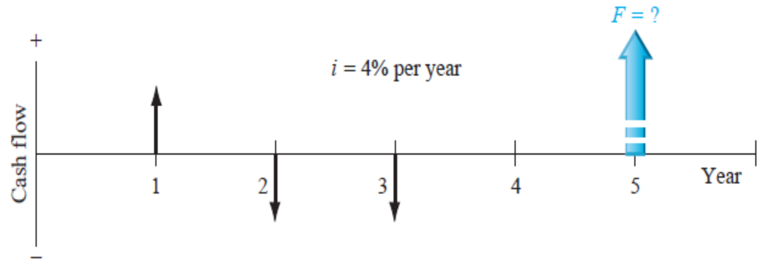


Figure (1-5): Example of positive and negative cash flows.

It is important to understand that future (F) and uniform annual (A) amounts are located at the end of the interest period, which is not necessarily December 31. Remember, end of the period means end of interest period, not end of calendar year.

The ***cash flow diagram*** is a very important tool in an economic analysis, especially when the cash flow series is complex. It is a graphical representation of cash flows drawn on the y axis with a time scale on the x axis. The diagram includes what is known, what is estimated, and what is needed. That is, once the cash flow diagram is complete, another person should be able to work the problem by looking at the diagram.

Cash flow diagram time $t = 0$ is the present, and $t = 1$ is the end of time period 1. We assume that the periods are in years for now. The time scale of Figure (1–4) is set up for 5 years. Since the end-of-year convention places cash flows at the ends of years, the "1" marks the end of year 1.

The direction of the arrows on the diagram is important to differentiate income from outgo.

- A vertical arrow pointing up indicates a positive cash flow.
- A down-pointing arrow indicates a negative cash flow.
- We will use a bold, colored arrow to indicate what is unknown and to be determined.

For example, if a future value F is to be determined in year 5, a wide, colored arrow with F= ? is shown in year 5. The interest rate is also indicated on the diagram. Figure (1–5) illustrates a cash inflow at the end of year 1, equal cash outflows at the end of years 2 and 3, an interest rate of 4% per year, and the unknown future value F after 5 years. The arrow for the unknown value is generally drawn in the opposite direction from the other cash flows; however, the engineering economy computations will determine the actual sign on the F value.

Before the diagramming of cash flows, a perspective or vantage point must be determined so that **+** or – signs can be assigned and the economic analysis performed correctly. Assume you borrow $8500 from a bank today to purchase an $8000 used car for cash next week, and you plan to spend the remaining $500 on a new paint job for the car two weeks from now. There are several perspectives possible when developing the cash flow diagram—those of the borrower (that's you), the banker, the car dealer, or the paint shop owner. The cash flow signs and amounts for these perspectives are as follows.

Perspective	*Activity*	*Cash flow with Sign, $*	*Time, week*
You	Borrow	+8500	0
	Buy car	-8000	1
	Paint job	-500	2
Banker	Lender	-8500	0
Car dealer	Car sale	+8000	1
Painter	Paint job	+500	2

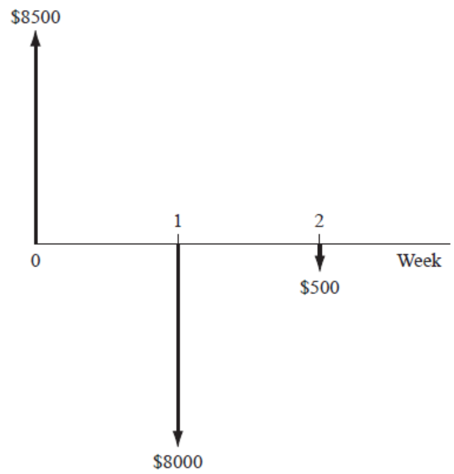


Figure (1-6): Cash flows from perspective of borrower for loan and purchases.

One, and only one, of the perspectives is selected to develop the diagram. For your perspective, all three cash flows are involved and the diagram appears as shown in Figure (1–6) with a time scale of weeks. Applying the end-of-period convention, you have a receipt of +$8500 now (time 0) and cash outflows of -$8000 at the end of week 1, followed by -$500 at the end of week 2.

Example (1-8)

Each year Exxon-Mobil expends large amounts of funds for mechanical safety features throughout its worldwide operations. Carla Ramos, a lead engineer for Mexico and Central American operations, plans expenditures of $1 million *now* and each of the next 4 years just for the improvement of field-based pressure-release valves. Construct the cash flow diagram to find the equivalent value of these expenditures at the end of year 4, using a cost of capital estimate for safety-related funds of 12% per year.

Solution

Figure (1–7) indicates the uniform and negative cash flow series (expenditures) for five periods, and the unknown *F* value (positive cash flow equivalent) at exactly the same time as the fifth expenditure. Since the expenditures start immediately, the first $1 million is shown at time 0, not time 1. Therefore, the last negative cash flow occurs at the end of the fourth year, when *F* also occurs. To make this diagram have a full 5 years on the time scale, the addition of the year -1 completes the diagram. This addition demonstrates that year 0 is the end-of-period point for the year -1.

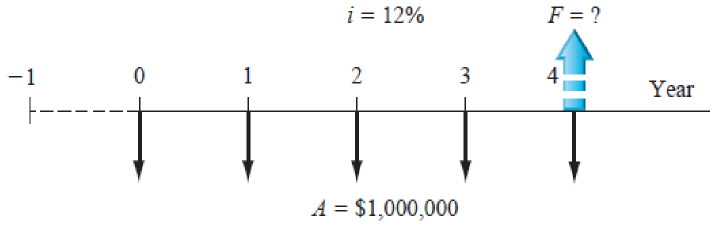


Figure (1-7): Cash flow diagram, Example (1-8).

Example (1-9)

An electrical engineer wants to deposit an amount *P* now such that she can withdraw an equal annual amount of $A1$ = $2000 per year for the first 5 years, starting 1 year after the deposit, and a different annual withdrawal of $A2$ = $3000 per year for the following 3 years. How would the cash flow diagram appear if i= 8.5% per year?

Solution

The cash flows are shown in Figure (1–8). The negative cash outflow *P* occurs now. The withdrawals (positive cash inflow) for the $A1$ series occur at the end of years 1 through 5, and $A2$ occurs in years 6 through 8.

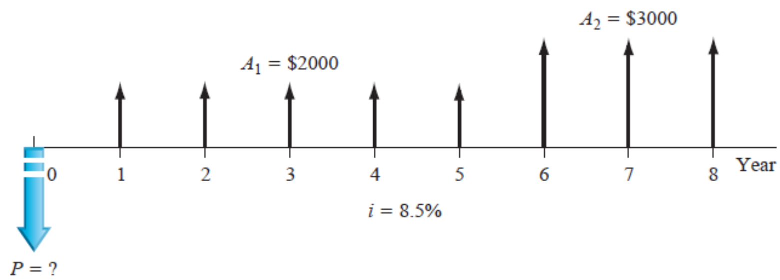


Figure (1-9): Cash flow diagram with two different A series, Example (1-9).

6. Economic Equivalence

Economic equivalence is a fundamental concept upon which engineering economy computations are based. Before we delve into the economic aspects, think of the many types of equivalency we may utilize daily by transferring from one scale to another. Some example transfers between scales are as follows:

Length:

12 inches = 1 foot 3 feet = 1 yard 39.370 inches = 1 meter
100 centimeters = 1 meter 1000 meters = 1 kilometer 1 kilometer = 0.621 mile

Pressure:

1 atmosphere =1 newton/meter ^{2} = 10 ^{3} pascal = 1 kilopascal

Speed:

1 mile = 1.609 kilometers 1 hour = 60 minutes
110 kilometers per hour (kph) = 68.365 miles per hour (mph)
68.365 mph = 1.139 miles per minute

Economic equivalence is a combination of interest rate and time value of money to determine the different amounts of money at different points in time that are equal in economic value.

As an illustration, if the interest rate is 6% per year, \$100 today (present time) is equivalent to \$106 one year from today.

$$Amount\ accrued = 100 + 100(0.06) = 100(1 + 0.06) = \$106$$

If someone offered you a gift of \$100 today or \$106 one year from today, it would make no difference which offer you accepted from an economic perspective. In either case you have \$106 one year from today. However, the two sums of money are equivalent to each other *only* when the interest rate is 6% per year. At a higher or lower interest rate, \$100 today is not equivalent to \$106 one year from today.

In addition to future equivalence, we can apply the same logic to determine equivalence for previous years. A total of \$100 now is equivalent to \$100/1.06 = \$94.34 one year ago at an interest rate of 6% per year. From these illustrations, we can state the following: \$94.34 last year, \$100 now, and \$106 one year from now are equivalent at an interest rate of 6% per year. The fact that these sums are equivalent can be verified by computing the two interest rates for 1-year interest periods.

$$\frac{\$6}{\$100} \times 100\% = 6\%\ per\ year\ \ and\ \ \frac{\$5.66}{\$94.34} \times 100\% = 6\%\ per\ year$$

The cash flow diagram in Figure (1–10) indicates the amount of interest needed each year to make these three different amounts equivalent at 6% per year.

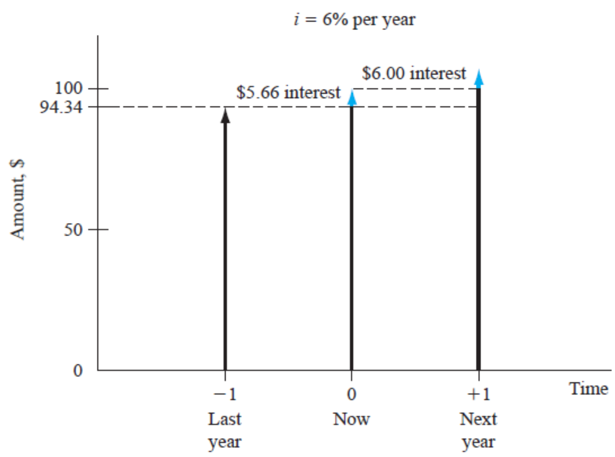


Figure (1-10): Equivalence of money at 6% per year interest.

Example (1-10)

Manufacturers make backup batteries for computer systems available to Batteries+ dealers through privately owned distributorships. In general, batteries are stored throughout the year, and a 5% cost increase is added each year to cover the inventory carrying charge for the distributorship owner. Assume you own the City Center Batteries+ outlet. Make the calculations necessary to show which of the following statements are true and which are false about battery costs.

1) The amount of \$98 now is equivalent to a cost of \$105.60 one year from now.

2) A truck battery cost of \$200 one year ago is equivalent to \$205 now.
3) A \$38 cost now is equivalent to \$39.90 one year from now.
4) A \$3000 cost now is equivalent to \$2887.14 one year earlier.
5) The carrying charge accumulated in 1 year on an investment of \$20,000 worth of batteries is \$1000.

Solution

1) Total amount accrued = 98(1.05) = \$102.90 ≠ \$105.60; therefore, it is false. Another way to solve this is as follows: Required original cost is 105.60/1.05 = \$100.57 ≠ \$98.
2) Equivalent cost 1 year ago is 205.00/1.05 = \$195.24 ≠ \$200; therefore, it is false.
3) The cost 1 year from now is \$38(1.05) = \$39.90; true.
4) Cost now is 2887.14(1.05) = \$3031.50 ≠ \$3000; false.
5) The charge is 5% per year interest, or \$20,000(0.05) = \$1000; true.

7. Simple and Compound Interest

Simple interest:

Is calculated using the principal only, ignoring any interest accrued in preceding interest periods. The total simple interest over several periods is computed as:

$$\boldsymbol{Simple\ interest = (principal)(number\ of\ periods)(interest\ rate)} \qquad (1-7)$$
$$\boldsymbol{I = Pni}$$

where *I* is the amount of interest earned or paid and the interest rate *i* is expressed in decimal form.

Example (1-11)

GreenTree Financing lent an engineering company \$100,000 to retrofit an environmentally unfriendly building. The loan is for 3 years at 10% per year simple interest. How much money will the fi rm repay at the end of 3 years?

Solution

The interest for each of the 3 years is

Interest per year = \$100,000(0.10) = \$10,000

Total interest for 3 years from Equation (1-7) is

Total interest = \$100,000(3)(0.10) = \$30,000

The amount due after 3 years is

Total due = \$100,000 + 30,000 = \$130,000

The interest accrued in the first year and in the second year does not earn interest. The interest due each year is \$10,000 calculated only on the \$100,000 loan principal.

Compound interest:

In most financial and economic analyses, we use compound interest calculations. For compound interest, the interest accrued for each interest period is calculated on the principal plus the total amount of interest accumulated in all previous periods. Thus, compound interest means interest on top of interest.

Compound interest reflects the effect of the time value of money on the interest also. Now the interest for one period is calculated as:

$$\textbf{Compound interest} = (\textbf{principal} = \textbf{all accrued interest})(\textbf{interest rate}) \quad (1-8)$$

In mathematical terms, the interest I_t for time period t may be calculated using the relation.

$$\boldsymbol{I_t} = \left(\boldsymbol{P} + \sum_{\boldsymbol{j=1}}^{\boldsymbol{j=t-1}} \boldsymbol{I_j}\right)(\boldsymbol{i}) \qquad (1-9)$$

Example (1-12)

Assume an engineering company borrows \$100,000 at 10% per year compound interest and will pay the principal and all the interest after 3 years. Compute the annual interest and total amount due after 3 years. Graph the interest and total owed for each year and compare with the previous example that involved simple interest.

Solution

To include compounding of interest, the annual interest and total owed each year are calculated by Equation (1-8).

Interest, year 1:	100,000(0.10) = \$10,000
Total due, year 1:	100,000 + 10,000 = \$110,000
Interest, year 2:	110,000(0.10) = \$11,000
Total due, year 2:	110,000 + 11,000 = \$121,000
Interest, year 3:	121,000(0.10) = \$12,100
Total due, year 3:	121,000 + 12,100 = \$133,100

The repayment plan requires no payment until year 3 when all interest and the principal, a total of \$133,100, are due. Figure (1–11) uses a cash flow diagram format to compare end-of-year *(a)* simple and *(b)* compound interest and total amounts owed. The differences due to compounding are clear. An extra \$133,100 – 130,000 = \$3100 in interest is due for the compounded interest loan.

Note that while simple interest due each year is constant, the compounded interest due grows geometrically. Due to this geometric growth of compound interest, the difference between simple and compound interest accumulation increases rapidly as the time frame increases. For example, if the loan is for 10 years, not 3, the extra paid for compounding interest may be calculated to be \$59,374.

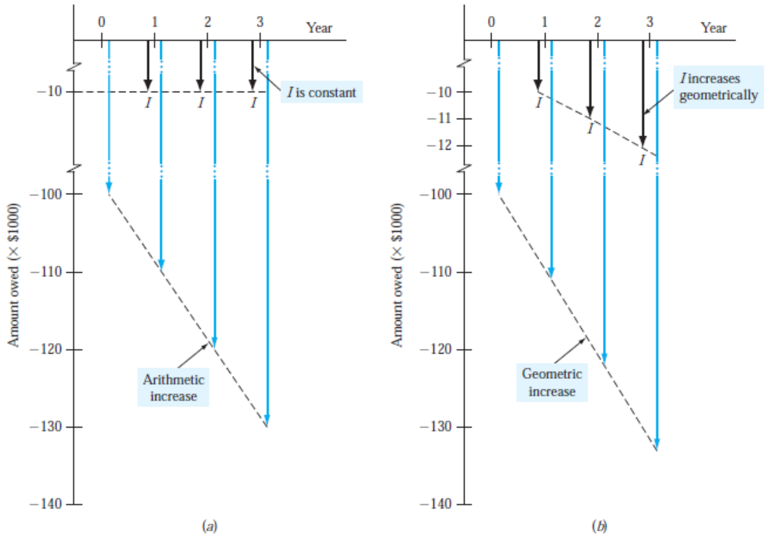


Figure (1-11): Interest I owed and total amount owed for (a) simple interest (Example (1-11)) and (b) compound interest (Example (1-12)).

A more efficient way to calculate the total amount due after a number of years in Example (1-12) is to utilize the fact that compound interest increases geometrically. This allows us to skip the year-by-year computation of interest. In this case, the ***total amount due at the end of each year*** is

Year 1: $\$100{,}000(1.10)^1 = \$110{,}000$

Year 2: $\$100{,}000(1.10)^2 = \$121{,}000$

Year 3: $\$100{,}000(1.10)^3 = \$133{,}100$

This allows future totals owed to be calculated directly without intermediate steps. The general form of the equation is:

$$\textbf{\textit{Total due after n years}} = \textbf{\textit{principal}}(\mathbf{1} + \textbf{\textit{interest rate}})^{\textbf{\textit{n years}}} \qquad (1-10)$$

$$= \boldsymbol{P}(\mathbf{1} + \boldsymbol{i})^{\boldsymbol{n}}$$

Where i is expressed in decimal form. Equation (1-10) was applied above to obtain the \$133,100 due after 3 years. This fundamental relation will be used many times in the upcoming chapters.

We can combine the concepts of interest rate, compound interest, and equivalence to demonstrate that different loan repayment plans may be equivalent but differ substantially in amounts paid from one year to another and in the total repayment amount. This also shows that there are many ways to take into account the time value of money.

Example (1-13)

Table (1–1) details four different loan repayment plans described below. Each plan repays a \$5000 loan in 5 years at 8% per year compound interest.

- **Plan 1: Pay all at end.** No interest or principal is paid until the end of year 5. Interest accumulates each year on the total of principal and all accrued interest.
- **Plan 2: Pay interest annually, principal repaid at end.** The accrued interest is paid each year, and the entire principal is repaid at the end of year 5.

- **Plan 3: Pay interest and portion of principal annually.** The accrued interest and one-fifth of the principal (or $1000) are repaid each year. The outstanding loan balance decreases each year, so the interest (column 2) for each year decreases.
- **Plan 4: Pay equal amount of interest and principal.** Equal payments are made each year with a portion going toward principal repayment and the remainder covering the accrued interest. Since the loan balance decreases at a rate slower than that in plan 3 due to the equal end-of-year payments, the interest decreases, but at a slower rate.

Table (1-1): Different Repayment Schedules Over 5 Years for $5000 at 8% Per Year Compound Interest

(1) End of Year	(2) Interest Owed for Year	(3) Total Owed at End of Year	(4) End-of-Year Payment	(5) Total Owed After Payment
Plan 1: Pay All at End				
0				$5000.00
1	$400.00	$5400.00	—	5400.00
2	432.00	5832.00	—	5832.00
3	466.56	6298.56	—	6298.56
4	503.88	6802.44	—	6802.44
5	544.20	7346.64	$ – 7346.64	
Total			$ – 7346.64	
Plan 2: Pay Interest Annually; Principal Repaid at End				
0				$5000.00
1	$400.00	$5400.00	$ – 400.00	5000.00
2	400.00	5400.00	– 400.00	5000.00
3	400.00	5400.00	– 400.00	5000.00
4	400.00	5400.00	– 400.00	5000.00
5	400.00	5400.00	– 5400.00	
Total			$ – 7000.00	
Plan 3: Pay Interest and Portion of Principal Annually				
0				$5000.00
1	$400.00	$5400.00	$ – 1400.00	4000.00
2	320.00	4320.00	–1320.00	3000.00
3	240.00	3240.00	–1240.00	2000.00
4	160.00	2160.00	– 1160.00	1000.00
5	80.00	1080.00	– 1080.00	
Total			$ – 6200.00	
Plan 4: Pay Equal Annual Amount of Interest and Principal				
0				$5000.00
1	$400.00	$5400.00	$ – 1252.28	4147.72
2	331.82	4479.54	– 1252.28	3227.25
3	258.18	3485.43	– 1252.28	2233.15
4	178.65	2411.80	– 1252.28	1159.52
5	92.76	1252.28	– 1252.28	
Total			$ – 6261.40	

1) Make a statement about the equivalence of each plan at 8% compound interest.
2) Develop an 8% per year simple interest repayment plan for this loan using the same approach as plan 2. Comment on the total amounts repaid for the two plans.

Solution

1) The amounts of the annual payments are different for each repayment schedule, and the total amounts repaid for most plans are different, even though each repayment plan requires exactly 5 years. The difference in the total amounts repaid can be explained by the time value of money and by the partial repayment of principal prior to year 5. A loan of $5000 at time 0 made at 8% per year compound interest is equivalent to each of the following:

- **Plan 1** $7346.64 at the end of year 5
- **Plan 2** $400 per year for 4 years and $5400 at the end of year 5
- **Plan 3** Decreasing payments of interest and partial principal in years 1 ($1400) through 5 ($1080)
- **Plan 4** $1252.28 per year for 5 years

An engineering economy study typically uses plan 4; interest is compounded, and a constant amount is paid each period. This amount covers accrued interest and a partial amount of principal repayment.

2) The repayment schedule for 8% per year simple interest is detailed in Table (1–2). Since the annual accrued interest of $400 is paid each year and the principal of $5000 is repaid in year 5, the schedule is exactly the same as that for 8% per year compound interest, and the total amount repaid is the same at $7000. In this unusual case, simple and compound interest result in the same total repayment amount. Any deviation from this schedule will cause the two plans and amounts to differ.

Table (1-2): A 5-Year Repayment Schedule of $5000 at 8% per Year Simple Interest

End of Year	Interest Owed for Year	Total Owed at End of Year	End-of-Year Payment	Total Owed After Payment
0				$5000
1	$400	$5400	$ – 400	5000
2	400	5400	– 400	5000
3	400	5400	– 400	5000
4	400	5400	– 400	0
5	400	5400	– 5400	
Total			$ – 7000	

8. Minimum Attractive Rate of Return

Engineering alternatives are evaluated upon the prognosis that a reasonable ROR can be expected. Therefore, some reasonable rate must be established for the selection criteria (step 4) of the engineering economy study (Figure(1–1)).

The Minimum Attractive Rate of Return (MARR) is a reasonable rate of return established for the evaluation and selection of alternatives. A project is not economically viable unless it is expected to return at least the MARR. MARR is also referred to as the hurdle rate, cutoff rate, benchmark rate, and minimum acceptable rate of return.

Figure (1–12) indicates the relations between different rate of return values. In the United States, the current U.S. Treasury Bill return is sometimes used as the benchmark safe rate. The MARR will always be higher than this, or a similar, safe rate. The MARR is not a rate that is calculated as a ROR. The MARR is established by (financial) managers and is used as a criterion against which an alternative's ROR is measured, when making the accept/reject investment decision.

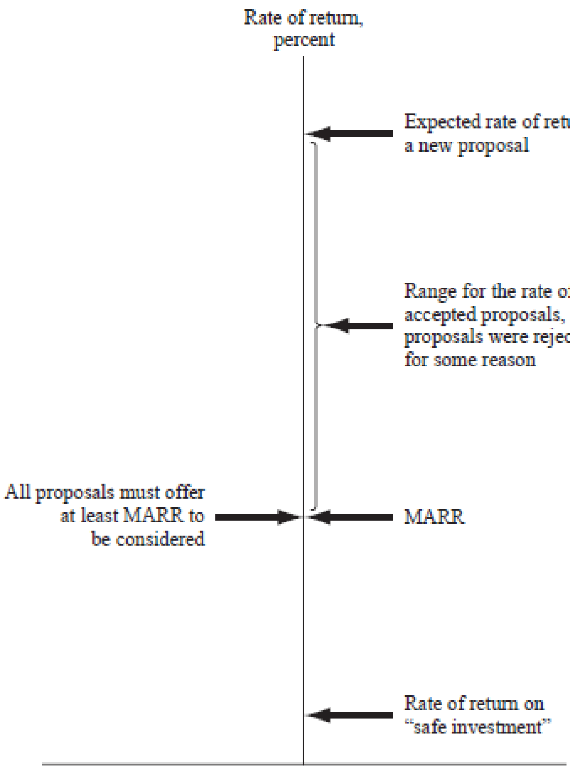


Figure (1-12): Size of MAAR relative to other rate of return values.

In general, capital is developed in two ways—equity financing and debt financing. A combination of these two is very common for most projects.

Equity financing: The corporation uses its own funds from cash on hand, stock sales, or retained earnings. Individuals can use their own cash, savings, or investments. In the example above, using money from the 5% savings account is equity financing.

Debt financing: The corporation borrows from outside sources and repays the principal and interest according to some schedule, much like the plans in Table (1–1). Sources of debt capital may be bonds, loans, mortgages, venture capital pools, and many others. Individuals, too, can utilize debt sources, such as the credit card (15% rate) and bank options (9% rate) described above.

Combinations of debt-equity financing mean that a ***weighted average cost of capital (WACC)*** results. If the HDTV is purchased with 40% credit card money at 15% per year and 60% savings account funds earning 5% per year, the weighted average cost of capital is

$$0.4(15) + 0.6(5) = 9\% \text{ per year.}$$

For a corporation, the established MARR used as a criterion to accept or reject an investment alternative will usually be equal to or higher than the WACC that the corporation must bear to obtain the necessary capital funds. So, the inequality

$$\boldsymbol{ROR \geq MARR > WACC} \qquad (1-11)$$

must be correct for an accepted project. Exceptions may be government-regulated requirements (safety, security, environmental, legal, etc.), economically lucrative ventures expected to lead to other opportunities, etc.

Often there are many alternatives that are expected to yield a ROR that exceeds the MARR as indicated in Figure (1–12), but there may not be sufficient capital available for all, or the project's risk may be estimated as too high to take the investment chance. Therefore, new projects that are undertaken usually have an expected return at least as great as the return on another alternative that is not funded. The expected rate of return on the unfunded project is called the ***opportunity cost.***

> ***The opportunity cost is the rate of return of a forgone opportunity caused by the inability to pursue a project. Numerically, it is the largest rate of return of all the projects not accepted (forgone) due to the lack of capital funds or other resources. When no specific MARR is established, the de facto MARR is the opportunity cost, i.e., the ROR of the first project not undertaken due to unavailability of capital funds.***

9. Spreadsheets use in engineering economy

The functions on a computer spreadsheet can greatly reduce the amount of hand work for equivalency computations involving *compound interest* and the terms *P* , *F* , *A* , *i* , and *n* . The use of a calculator to solve most simple problems is preferred by many students and professors. However, as cash flow series become more complex, the spreadsheet offers a good alternative. Microsoft Excel is used throughout this book because it is readily available and easy to use.

A total of seven Excel functions can perform most of the fundamental engineering economy calculations. The functions are great supplemental tools, but they do not replace the understanding of engineering economy relations, assumptions, and techniques. Using the symbols *P* , *F* , *A* , *i* , and *n* defined in the previous section, the functions most used in engineering economic analysis are formulated as follows.

- To find the present value *P:* = **PV(*i%, n, A, F*)**
- To find the future value *F:* = **FV(*i%, n, A, P*)**
- To find the equal, periodic value *A:* = **PMT(*i%, n, P, F*)**
- To find the number of periods *n:* = **NPER(*i%, A, P, F*)**
- To find the compound interest rate *i:* = **RATE(*n, A, P, F*)**
- To find the compound interest rate *i:* = **IRR(fi rst_cell:last_cell)**
- To find the present value *P* of any series: = **NPV(i %, second_cell:last_cell) + fi rst_cell**

If some of the parameters don't apply to a particular problem, they can be omitted and zero is assumed. For readability, spaces can be inserted between parameters within parentheses. If the parameter omitted is an interior one, the comma must be entered. The last two functions require that a series of numbers be entered into contiguous spreadsheet cells, but the first five can be used with no supporting data. In all cases, the function must be preceded by an equals sign (+) in the cell where the answer is to be displayed.

Example (1-14)

A Japan-based architectural fi rm has asked a United States–based software engineering group to infuse GPS sensing capability via satellite into monitoring software for high-rise structures in order to detect greater than expected horizontal movements. This software could be very beneficial as an advance warning of serious tremors in earthquake-prone areas in Japan and the United States. The inclusion of accurate GPS data is estimated to increase annual revenue over that for the current software system by \$200,000 for each of the next 2 years, and by \$300,000 for each of years 3 and 4. The planning horizon is only 4 years due to the rapid advances made internationally in building-monitoring software. Develop spreadsheets to answer the questions below.

1) Determine the total interest and total revenue after 4 years, using a compound rate of return of 8% per year.
2) Repeat part (1) if estimated revenue increases from \$300,000 to \$600,000 in years 3 and 4.
3) Repeat part (1) if inflation is estimated to be 4% per year. This will decrease the real rate of return from 8% to 3.85% per year.

Solution

Refer to Figure (1–13) *a* to *d* for the solutions. All the spreadsheets contain the same information, but some cell values are altered as required by the question. (Actually, all the questions can be answered on one spreadsheet by changing the numbers. Separate spreadsheets are shown here for explanation purposes only.)

The Excel functions are constructed with reference to the cells, not the values themselves, so that sensitivity analysis can be performed without function changes. This approach treats the value in a cell as a *global variable* for the spreadsheet. For example, the 8% rate in cell B2 will be referenced in all functions as B2, not 8%. Thus, a change in the rate requires only one alteration in the cell B2 entry, not in every relation where 8% is used.

1) Figure (1–13 a) shows the results, and Figure (1–13 b) presents all spreadsheet relations for estimated interest and revenue (yearly in columns C and E, cumulative in columns D and F). As an illustration, for year 3 the interest I3 and revenue plus interest R3 are

$$
\begin{aligned}
I_3 &= \text{(cumulative revenue through year 2) (rate of return)} \\
&= \$416{,}000(0.08) \\
&= \$33{,}280 \\
R_3 &= \text{revenue in year 3} + I_3 \\
&= \$300{,}000 + 33{,}280 \\
&= \$333{,}280
\end{aligned}
$$

The detailed relations shown in Figure (1–13 *b)* calculate these values in cells C8 and E8.

Cell C8 relation for I_3 : = F7×B2

Cell E8 relation for CF_3 := B8 + C8

The equivalent amount after 4 years is \$1,109,022, which is comprised of \$1,000,000 in total revenue and \$109,022 in interest compounded at 8% per year. The shaded cells in Figure (1–13 a) and b indicate that the sum of the annual values and the last entry in the cumulative columns must be equal.

2) To determine the effect of increasing estimated revenue for years 3 and 4 to \$600,000, use the same spreadsheet and change the entries in cells B8 and B9 as shown in Figure (1–13 c). Total interest increases 22%, or \$24,000, from \$109,222 to \$133,222.
3) Figure (1–13 d) shows the effect of changing the original i value from 8% to an inflation adjusted rate of 3.85% in cell B2 on the first spreadsheet. [Remember to return to the \$300,000 revenue estimates for years 3 and 4 after working part (2).] Inflation has now reduced total interest by 53% from \$109,222 to \$51,247, as shown in cell C10.

	A	B	C	D	E	F
1	Part (a) - Find totals in year 4					
2	i =	8.0%				
3-4	End of Year	Revenue at end of year, $	Interest earned during year, $	Cumulative interest, $	Revenue during year with interest, $	Cumulative revenue with interest, $
5	0					
6	1	200,000	0	0	200,000	200,000
7	2	200,000	16,000	16,000	216,000	416,000
8	3	300,000	33,280	49,280	333,280	749,280
9	4	300,000	59,942	109,222	359,942	1,109,222
10			109,222		1,109,222	

Figure (1-13 a): Total interest and revenue for base case, year 4

	A	B	C	D	E	F
1	Part (a) - Find totals in year 4					
2	i =	0.08				
3-4	End of Year	Revenue at end of year, $	Interest earned during year, $	Cumulative interest, $	Revenue during year with interest, $	Cumulative revenue with interest, $
5	0					
6	1	200000	0	=C6	=B6 + C6	=E6
7	2	200000	=F6*B1	=C7 + D6	=B7 + C7	=E7 + F6
8	3	300000	=F7*B1	=C8 + D7	=B8 + C8	=E8 + F7
9	4	300000	=F8*B1	=C9 + D8	=B9 + C9	=E9 + F8
10			=SUM(C6:C9)		=SUM(E6:E9)	

Figure (1-13 b): Spreadsheet relations for base case

	A	B	C	D	E	F
1	Part (b) - Find totals in year 4 with increased revenues					
2	i =	8.0%				
3-4	End of Year	Revenue at end of year, $	Interest earned during year, $	Cumulative interest, $	Revenue during year with interest, $	Cumulative revenue with interest, $
5	0					
6	1	200,000	0	0	200,000	200,000
7	2	200,000	16,000	16,000	216,000	416,000
8	3	600,000	33,280	49,280	633,280	1,049,280
9	4	600,000	83,942	133,222	683,942	1,733,222
10			133,222		1,733,222	
11						
12			Revenue changed			
13						

Figure (1-13 c): Totals with increased revenue in years 3 and 4

Rate of return changed

	A	B	C	D	E	F
1	Part (c) - Find totals in year 4 considering 4% inflation					
2	i =	3.85%				
3-4	End of Year	Revenue at end of year, $	Interest earned during year, $	Cumulative interest, $	Revenue during year with interest, $	Cumulative revenue with interest, $
5	0					
6	1	200,000	0	0	200,000	200,000
7	2	200,000	7,700	7,700	207,700	407,700
8	3	300,000	15,696	23,396	315,696	723,396
9	4	300,000	27,851	51,247	327,851	1,051,247
10			51,247		1,051,247	

Figure (1-13 d): Totals with inflation of 4% per year considered

Problems

(1-1) List the four essential elements involved in decision making in engineering economic analysis.

(1-2) What is meant by (a) limited capital funds and (b) sensitivity analysis?

(1-3) List three measures of worth that are used in engineering economic analysis.

(1-4) Identify the following factors as either economic (tangible) or noneconomic (intangible): first cost, leadership, taxes, salvage value, morale, dependability, inflation, profit, acceptance, ethics, interest rate.

(1-5) Emerson Processing borrowed $900,000 for installing energy-efficient lighting and safety equipment in its La Grange manufacturing facility. The terms of the loan were such that the company could pay interest only at the end of each year for up to 5 years, after which the company would have to pay the entire amount due. If the interest rate on the loan was 12% per year and the company paid only the interest for 4 years, determine the following:

a) The amount of each of the four interest payments
b) The amount of the final payment at the end of year 5

(1-6) Which of the following 1-year investments has the highest rate of return?

a) $12,500 that yields $1125 in interest,
b) $56,000 that yields $6160 in interest, or
c) $95,000 that yields $7600 in interest.

(1-7) The symbol P represents an amount of money at a time designated as present. The following symbols also represent a present amount of money and require similar calculations. Explain what each symbol stands for: PW, PV, NPV, DCF, and CC

(1-8) What is meant by end-of-period convention?

(1-9) Construct a cash flow diagram to find the present worth in year 0 at an interest rate of 15% per year for the following situation.

Year	***Cash Flow, $***
0	-19,000
1–4	+8,100

(1-10) Construct a cash flow diagram that represents the amount of money that will be accumulated in 15 years from an investment of $40,000 now at an interest rate of 8% per year.

(1-11) At an interest rate of 15% per year, an investment of $100,000 one year ago is equivalent to how much now?

(1-12) University tuition and fees can be paid by using one of two plans.
Early-bird: Pay total amount due 1 year in advance and get a 10% discount.
On-time: Pay total amount due when classes start.
The cost of tuition and fees is $10,000 per year.

a) How much is paid in the early-bird plan?
b) What is the equivalent amount of the savings compared to the on-time payment at the time that the on-time payment is made?

(1-13) If a company sets aside $1,000,000 now into a contingency fund, how much will the company have in 2 years, if it does not use any of the money and the account grows at a rate of 10% per year?

(1-14) To finance a new product line, a company that makes high-temperature ball bearings borrowed $1.8 million at 10% per year interest. If the company repaid the loan in a lump sum amount after 2 years, what was:

a) The amount of the payment

b) The amount of interest?

(1-15) If interest is compounded at 20% per year, how long will it take for \$50,000 to accumulate to \$86,400?

(1-16) Give three other names for minimum attractive rate of return.

(1-17) What is the weighted average cost of capital for a corporation that finances an expansion project using 30% retained earnings and 70% venture capital? Assume the interest rates are 8% for the equity financing and 13% for the debt financing.

(1-18) State the purpose for each of the following built-in spreadsheet functions.

a) *PV (i% , n , A , F)*
b) *FV (i %, n , A , P)*
c) *RATE (n , A , P , F)*
d) *IRR (first_cell:last_cell)*
e) *PMT (i %, n , P , F)*
f) *NPER (i %, A , P , F)*

(1-19) What are the values of the engineering economy symbols P , F , A , i , and n in the following functions? Use a question mark for the symbol that is to be determined.

a) *NPER (8%,-1500,8000,2000)*
b) *FV (7%,102000,-9000)*
c) *RATE (10,1000,-12000,2000)*
d) *PMT (11%,20,,14000)*
e) *PV (8%,15,-1000,800)*

Choose the correct answers for the following questions.

(1-1) The concept that different sums of money at different points in time can be said to be equal to each other is known as:
a) Evaluation criterion
b) Equivalence
c) Cash flow
d) Intangible factors

(1-2) The evaluation criterion that is usually used in an economic analysis is:
a) Time to completion
b) Technical feasibility
c) Sustainability
d) Financial units (dollars or other currency)

(1-3) All of the following are examples of cash outflows, except:
a) Asset salvage value
b) Income taxes
c) Operating cost of asset
d) First cost of asset

(1-4) In most engineering economy studies, the best alternative is the one that:
a) Will last the longest time
b) Is most politically correct
c) Is easiest to implement
d) Has the lowest cost

(1-5) The following annual maintenance and operation (M&O) costs for a piece of equipment were collected over a 5-year period: $12,300, $8900, $9200, $11,000, and $12,100. The average is $10,700. In conducting a sensitivity analysis, the most reasonable range of costs to use (i.e., percent from the average) is:
a) ±5%
b) ±11%
c) ±17%
d) ±25%

(1-6) At an interest rate of 10% per year, the equivalent amount of $10,000 one year ago is closest to:
a) $8264
b) $9091
c) $11,000
d) $12,000

(1-7) Assume that you and your best friend each have $1000 to invest. You invest your money in a fund that pays 10% per year compound interest. Your friend invests her money at a bank that pays 10% per year simple interest. At the end of 1 year, the difference in the total amount for each of you is:

a) You have $10 more than she does
b) You have $100 more than she does
c) You both have the same amount of money
d) She has $10 more than you do

(1-8) The time it would take for a given sum of money to double at 4% per year simple interest is closest to:

a) 30 years
b) 25 years
c) 20 years
d) 10 years

(1-9) All of the following are examples of equity financing, except :

a) Mortgage
b) Money from savings
c) Cash on hand
d) Retained earnings

(1-10) To finance a new project costing $30 million, a company borrowed $21 million at 16% per year interest and used retained earnings valued at 12% per year for the remainder of the investment. The company's weighted average cost of capital for the project was closest to:

a) 12.5%
b) 13.6%
c) 14.8%
d) 15.6%

Answers:

Question	***answer***
(1-1)	b
(1-2)	d
(1-3)	a
(1-4)	d
(1-5)	c
(1-6)	b
(1-7)	c
(1-8)	b
(1-9)	a
(1-10)	c

Chapter 2

Factors: How Time and Interest Affect Money

Chapter 2
Factors: How Time and Interest Affect Money

General Objective:

Trainee will be able to understand the basic concepts and terminology necessary for engineering economy

Detailed Objectives:

1. Single-Payment Factors (F/P and P/F).
2. Uniform Series Formulas (P/A, A/P, A/F, F/A).
3. Arithmetic Gradient Factors (P⁄G and A⁄G).
4. Geometric Gradient Series Factors.
5. Calculations for Cash Flows That Are Shifted.
6. Using Spreadsheets for Equivalency Computation.

Introduction

The cash flow is fundamental to every economic study. Cash flows occur in many configurations and amounts—isolated single values, series that are uniform, and series that increase or decrease by constant amounts or constant percentages. This chapter develops derivations for all the commonly used engineering economy factors that take the time value of money into account. The application of factors is illustrated using their mathematical forms and a standard notation format. Spreadsheet functions are used in order to rapidly work with cash flow series and to perform sensitivity analysis.

1. Single-Amount Factors (F / P and P / F)

The ***most fundamental factor in engineering economy*** is the one that determines the amount of money *F* accumulated after *n* years (or periods) from a single present worth *P*, with interest compounded one time per year (or period). Recall that compound interest refers to interest paid on top of interest. Therefore, if an amount P is invested at time $t = 0$, the amount F_1 accumulated 1 year hence at an interest rate of *i* percent per year will be

$$\boldsymbol{F_1 = P + Pi = P(1+i)}$$

where the interest rate is expressed in decimal form. At the end of the second year, the amount accumulated F_2 is the amount after year 1 plus the interest from the end of year 1 to the end of year 2 on the entire F_1.

$$\boldsymbol{F_2 = F_1 + F_1 i = P(1+i) + P(1+i)i} \qquad (2-1)$$

The amount F_2 can be expressed as

$$\boldsymbol{F_2 = P(1+i+i+i^2) = P(1+2i+i^2) = P(1+i)^2}$$

Similarly, the amount of money accumulated at the end of year 3, using Equation (2-1), will be

$$\boldsymbol{F_3 = F_2 + F_2 i}$$

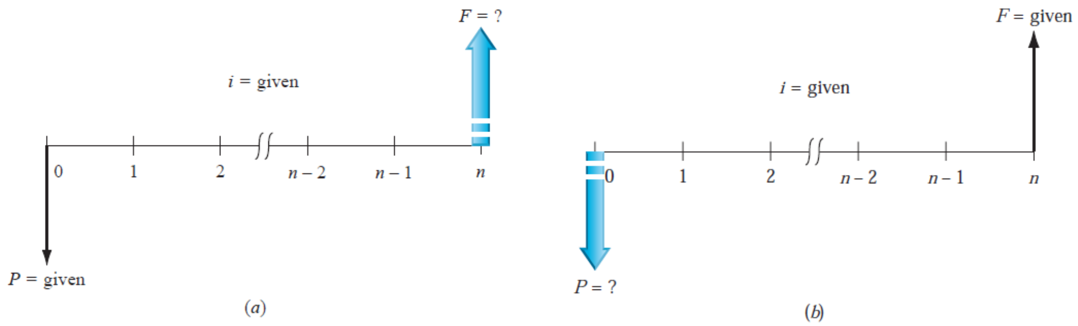


Figure (2-1): Cash flow diagrams for single-payment factors: (a) find F, given P, and (b) find P, given F.

From the preceding values, it is evident by mathematical induction that the formula can be generalized for *n* years. To find *F*, given *P*,

$$\boldsymbol{F = P(1+i)^n} \qquad (2-2)$$

The factor $(\boldsymbol{1+i})^{\boldsymbol{n}}$ is called the ***single-payment compound amount factor (SPCAF)***, but it is usually referred to as the ***F/P factor***. This is the conversion factor that, when multiplied by *P*, yields the future amount *F* of an initial amount *P* after *n* years at interest rate *i*. The cash flow diagram is seen in Figure (2–1 a).

Reverse the situation to ***determine the P value for a stated amount F*** that occurs *n* periods in the future. Simply solve Equation (2-2) for *P*.

$$\boldsymbol{P} = \boldsymbol{F}\left[\frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right] = \boldsymbol{F}(\mathbf{1}+\boldsymbol{i})^{-\boldsymbol{n}} \qquad (2-3)$$

The expression $(\mathbf{1}+\boldsymbol{i})^{-\boldsymbol{n}}$ is known as the ***single-payment present worth factor (SPPWF)***, or the ***P/F factor***. This expression determines the present worth *P* of a given future amount *F* after *n* years at interest rate *i*. The cash flow diagram is shown in Figure (2–1 b).

Note that the two factors derived here are for *single payments;* that is, they are used to find the present or future amount when only one payment or receipt is involved.

A standard notation has been adopted for all factors. The notation includes two cash flow symbols, the interest rate, and the number of periods. It is always in the general form (X/Y, i, n). The letter X represents what is sought, while the letter Y represents what is given. For example, F/P means find F when given P. The i is the interest rate in percent, and n represents the number of periods involved.

Using this notation, (F/*P* ,6%,20) represents the factor that is used to calculate the future amount *F* accumulated in 20 periods if the interest rate is 6% per period. The *P* is given. The standard notation, simpler to use than formulas and factor names, will be used hereafter.

Table (2–1) summarizes the standard notation and equations for the *F*/*P* and *P*/*F* factors.

Table (2–1): F/P and P/F Factors: Notation and Equations

Factor Notation	Factor Name	Find/Given	Standard Notation Equation	Equation with Factor Formula	Excel Function
$(F/P, i, n)$	Single-payment compound amount	F/P	$F = P(F/P, i, n)$	$F = P(1+i)^n$	= *FV*(*i*%,*n*,,*P*)
$(P/F, i, n)$	Single-payment present worth	P/F	$P = F(P/F, i, n)$	$P = F(1+i)^{-n}$	= *PV*(*i*%,*n*,,*F*)

For **spreadsheets**, a future value *F* is calculated by the FV function using the format

$$= \boldsymbol{FV}(\boldsymbol{i}\%, \boldsymbol{n},, \boldsymbol{P}) \qquad (2-4)$$

A present amount *P* is determined using the PV function with the format

$$= \boldsymbol{PV}(\boldsymbol{i}\%, \boldsymbol{n},, \boldsymbol{F}) \qquad (2-5)$$

Example (2-1)

Sandy, a manufacturing engineer, just received a year-end bonus of $10,000 that will be invested immediately. With the expectation of earning at the rate of 8% per year, Sandy hopes to take the entire amount out in exactly 20 years to pay for a family vacation when the oldest daughter is due to graduate from college. Find the amount of funds that will be available in 20 years by using:

1) hand solution by applying the factor formula and tabulated value
2) a spreadsheet functions.

Solution

The cash flow diagram is the same as Figure (2–1*a*). The symbols and values are

P = $10,000 *F* = ? *i* = 8% per year *n* = 20 years

1) Factor formula: Apply Equation (2-2) to find the future value F. Rounding to four decimals, we have

$$F = P(1+i)^n = 10{,}000(1.08)^{20} = 10{,}000(4.6610) = \$46{,}610$$

Standard notation and tabulated value: Notation for the F/P factor is (F/P,i%,n).

$$F = P(F/P, 8\%, 20) = 10{,}000(4.6610) = \$46{,}610$$

2) Spreadsheet: Use the FV function to find the amount 20 years in the future. The format is that shown in Equation (2-4); the numerical entry is *= FV(8%,20,,10000).*

2. Uniform Series Formulas (P/A, A/P, A/F, F/A)

The equivalent present worth P of a uniform series A of end-of-period cash flows (investments) is shown in Figure (2–2 a). An expression for the present worth can be determined by considering each A value as a future worth F, calculating its present worth with the P/F factor, Equation (2-3), and summing the results.

$$P = A\left[\frac{1}{(1+i)^1}\right] + A\left[\frac{1}{(1+i)^2}\right] + \cdots + A\left[\frac{1}{(1+i)^{n-1}}\right] + A\left[\frac{1}{(1+i)^n}\right] \qquad (2-6)$$

The terms in brackets are the P/F factors for years 1 through n, respectively. Factor out A.

To simplify Equation (2-6) and obtain the P/A factor, multiply the n -term geometric progression in brackets by the *(P/F, i %,1)* factor, which is *1/(1 +i)*. Simplify to obtain the expression for P when i ≠ 0 (Equation (2-7)).

$$\boldsymbol{P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] ; i \neq 0} \qquad (2-7)$$

The term in brackets in Equation (2-7) is the conversion factor referred to as the *uniform series present worth factor* (USPWF). It is the ***P*/*A* factor** used to calculate the ***equivalent P value in year 0*** for a uniform end-of-period series of A values beginning at the end of period 1 and extending for n periods. The cash flow diagram is Figure (2–2 a).

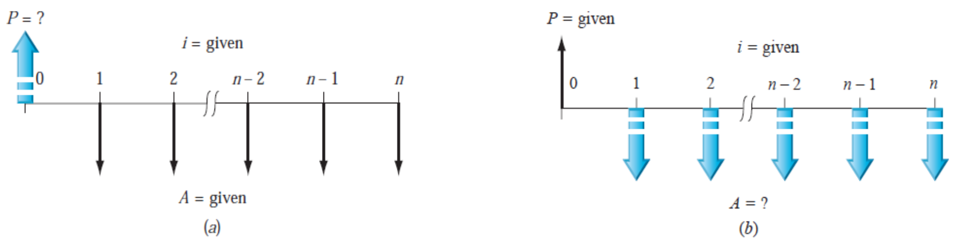


Figure (2-2): Cash flow diagrams used to determine: (a) P, given a uniform series A, and (b) A, given a present worth P.

To reverse the situation, the present worth P is known and the equivalent uniform series amount A is sought (Figure (2–4 b)). The first A value occurs at the end of period 1, that is, one period after P occurs. Solve Equation (2-7) for A to obtain

$$\boldsymbol{A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]} \qquad (2-8)$$

The term in brackets is called the *capital recovery factor* (CRF), or ***A*/*P* factor**. It calculates the ***equivalent uniform annual worth A*** over n years for a given P in year 0, when the interest rate is i.

The P/A and A/P factors are derived with the present worth P and the first uniform annual amount A one year (period) apart. That is, the present worth P must always be located one period prior to the first A

Spreadsheet functions can determine both *P* and *A* values in lieu of applying the *P/A* and *A/P* factors. The PV function calculates the *P* value for a given *A* over *n* years and a separate *F* value in year *n*, if it is given. The format, is

$$= \boldsymbol{PV}(\,\boldsymbol{i}\,\%, \boldsymbol{n}\,, \boldsymbol{A}\,, \boldsymbol{F}\,) \qquad (2-9)$$

Similarly, the *A* value is determined by using the PMT function for a given *P* value in year 0 and a separate *F*, if given. The format is

$$= \boldsymbol{PMT}(\,\boldsymbol{i}\,\%, \boldsymbol{n}\,, \boldsymbol{P}\,, \boldsymbol{F}\,) \qquad (2-10)$$

The simplest way to derive the *A/ F* factor is to substitute into factors already developed. If *P* from Equation (2-3) is substituted into Equation (2-8), the following formula results.

$$\boldsymbol{A} = \boldsymbol{F}\left[\frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right]\left[\frac{\boldsymbol{i}(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\mathbf{1}}\right]\;;\;\; \boldsymbol{A} = \boldsymbol{F}\left[\frac{\boldsymbol{i}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\mathbf{1}}\right] \qquad (2-11)$$

The expression in brackets in Equation (2-11) is the ***A/F* or sinking fund factor**. It determines the **uniform annual series *A*** that is equivalent to a given future amount *F*. This is shown graphically in Figure (2–3 a), where *A* is a uniform annual investment.

The uniform series A begins at the end of year (period) 1 and continues through the year of the given F. The last A value and F occur at the same time.

Equation (2-11) can be rearranged to find *F* for a stated *A* series in periods 1 through *n* (Figure 2–3 b).

$$\boldsymbol{F} = \boldsymbol{A}\left[\frac{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\mathbf{1}}{\boldsymbol{i}}\right] \qquad (2-12)$$

For solution by **spreadsheet**, the FV function calculates *F* for a stated *A* series over *n* years. The format is

$$= \boldsymbol{FV}(\,\boldsymbol{i}\,\%, \boldsymbol{n}\,, \boldsymbol{A}\,, \boldsymbol{P}\,) \qquad (2-13)$$

The *P* may be omitted when no separate present worth value is given. The PMT function determines the *A* value for *n* years, given *F* in year *n* and possibly a separate *P* value in year 0. The format is

$$= \boldsymbol{PMT}(\,\boldsymbol{i}\,\%, \boldsymbol{n}\,, \boldsymbol{P}\,, \boldsymbol{F}\,) \qquad (2-14)$$

If *P* is omitted, the comma must be entered so the function knows the last entry is an *F* value.

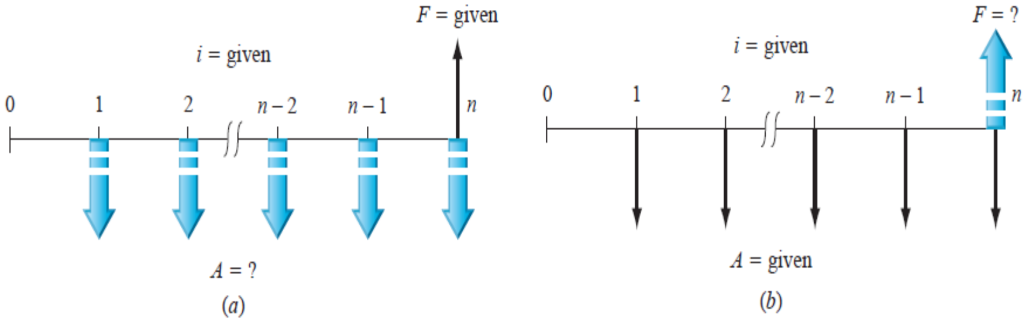


Figure (2-3): Cash flow diagrams to (a) find A, given F, and (b) find F, given A.

Table (2–2) summarizes the standard notation and equations for the *A/P*, *P/A*, *F/A* and *A/F* factors.

Table (2-2): P/A, A/P, A/F and F/A Factors: Notation and Equations

Notation	Factor Name	Find/Given	Factor Formula	Standard Notation Equation	Excel Function
$(P/A, i, n)$	Uniform series present worth	P/A	$\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]$	$P = A(P/A, i, n)$	= *PV*(*i*%,*n*,*A*)
$(A/P, i, n)$	Capital recovery	A/P	$\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$	$A = P(A/P, i, n)$	= *PMT*(*i*%,*n*,*P*)
$(F/A, i, n)$	Uniform series compound amount	F/A	$\left[\frac{(1+i)^n-1}{i}\right]$	$F = A(F/A, i, n)$	= *FV*(*i*%,*n*,*A*)
$(A/F, i, n)$	Sinking fund	A/F	$\left[\frac{i}{(1+i)^n-1}\right]$	$A = F(A/F, i, n)$	= *PMT*(*i*%,*n*,*F*)

Example (2-2)

How much money should you be willing to pay now for a guaranteed $600 per year for 9 years starting next year, at a rate of return of 16% per year?

Solution

The cash flows follow the pattern of Figure 2–2 *a* , with A = \$600, i = 16%, and n = 9. The present worth is

$$P = 600(P/A, 16\%, 9) = 600(4.6065) = \$2763.90$$

The PV function = *PV(16%,9,600)* entered into a single spreadsheet cell will display the answer P = (\$2763.93).

Example (2-3)

The president of Ford Motor Company wants to know the equivalent future worth of a $1 million capital investment each year for 8 years, starting 1 year from now. Ford capital earns at a rate of 14% per year.

Solution

The cash flow diagram (Figure (2–4)) shows the annual investments starting at the end of year 1 and ending in the year the future worth is desired. In \$1000 units, the F value in year 8 is found by using the F/A factor.

$$F = 1000(F/A, 14\%, 8) = 1000(13.2328) = \$13,232.80$$

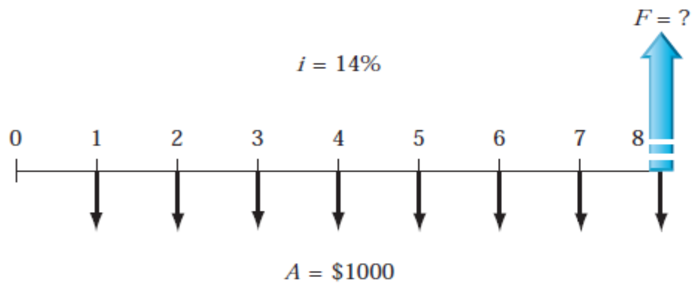


Figure (2–4): Diagram to find F for a uniform series, Example (2-3).

3. Arithmetic Gradient Factors (P/G and A/G)

Assume a manufacturing engineer predicts that the cost of maintaining a robot will increase by $5000 per year until the machine is retired. The cash flow series of maintenance costs involves a constant gradient, which is $5000 per year.

An arithmetic gradient series is a cash flow series that either increases or decreases by a constant amount each period. The amount of change is called the gradient.

Formulas previously developed for an *A* series have year-end amounts of equal value. In the case of a gradient, each year-end cash flow is different, so new formulas must be derived. First, assume that the cash flow at the end of year 1 is the **base amount** of the cash flow series and, therefore, not part of the gradient series. This is convenient because in actual applications, the base amount is usually significantly different in size compared to the gradient. For example, if you purchase a used car with a 1-year warranty, you might expect to pay the gasoline and insurance costs during the first year of operation. Assume these costs $2500; that is, $2500 is the base amount. After the first year, you absorb the cost of repairs, which can be expected to increase each year. If you estimate that total costs will increase by $200 each year, the amount the second year is $2700, the third $2900, and so on to year *n*, when the total cost is 2500 + (*n* - 1)200. The cash flow diagram is shown in Figure (2–5). Note that the gradient ($200) is first observed between year 1 and year 2, and the base amount ($2500 in year 1) is not equal to the gradient.

Define the symbols *G* for gradient and CF_n for cash flow in year n as follows.

G = constant arithmetic change in cash flows from one time period to the next; G may be positive or negative.

$$\boldsymbol{CF_n = base\ amount + (n-1)\,G} \qquad (2-15)$$

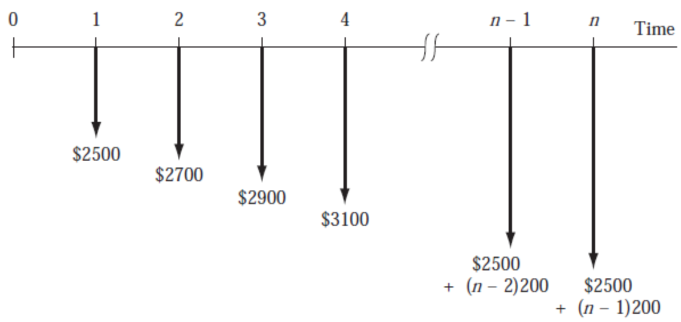


Figure (2–5): Cash flow diagram of an arithmetic gradient series.

It is important to realize that the base amount defines a uniform cash flow series of the size *A* that occurs each time period. We will use this fact when calculating equivalent amounts that involve arithmetic gradients. If the base amount is ignored, a generalized arithmetic (increasing) gradient cash flow diagram is as shown in Figure (2–6). Note that the gradient begins between years 1 and 2. This is called a **conventional gradient.**

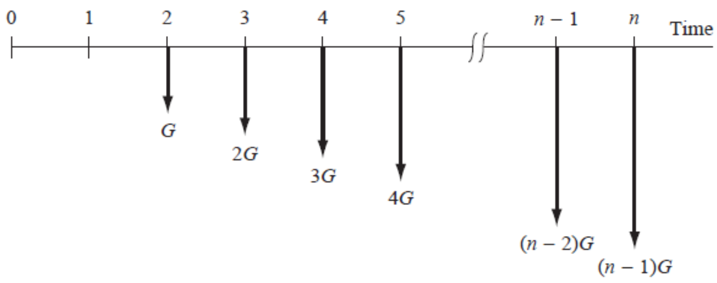


Figure (2–6): Conventional arithmetic gradient series without the base amount.

Example (2-4)

A local university has initiated a logo-licensing program with the clothier Holister, Inc. Estimated fees (revenues) are $80,000 for the first year with uniform increases to a total of $200,000 by the end of year 9. Determine the gradient and construct a cash flow diagram that identifies the base amount and the gradient series.

Solution

The year 1 base amount is CF_1 = $80,000, and the total increase over 9 years is

$$CF_9 - CF_1 = 200{,}000 - 80{,}000 = \$120{,}000$$

Equation (2-15), solved for *G*, determines the arithmetic gradient.

$$G = \left[\frac{CF_9 - CF_1}{n-1}\right] = \left[\frac{\$120{,}000}{9-1}\right] = \$15{,}000 \text{ per year}$$

The cash flow diagram Figure (2–7) shows the base amount of $80,000 in years 1 through 9 and the $15,000 gradient starting in year 2 and continuing through year 9.

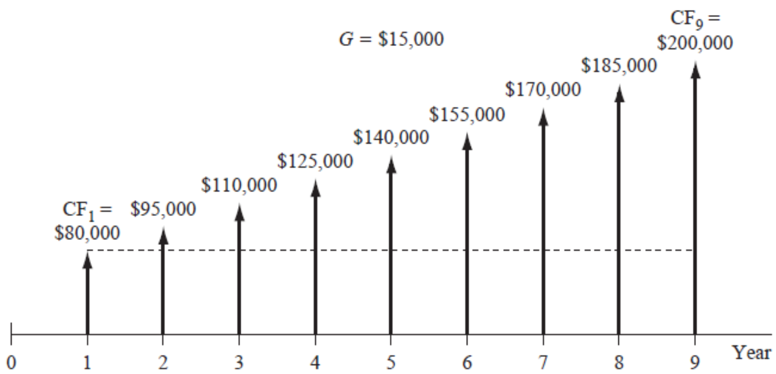


Figure (2–7): Diagram for gradient series, Example (2-4).

The **total present worth P_T** for a series that includes a base amount *A* and conventional arithmetic gradient must consider the present worth of both the uniform series defined by *A* and the arithmetic gradient series. The addition of the two results in P_T.

$$\boldsymbol{P_T = P_A \pm P_G} \qquad (2-16)$$

where P_A is the present worth of the uniform series only, P_G is the present worth of the gradient series only, and the – or + sign is used for an increasing (–*G)* or decreasing (+*G)* gradient, respectively.

The corresponding equivalent annual worth A_T is the sum of the base amount series annual worth A_A and gradient series annual worth A_G, that is,

$$A_T = A_A \pm A_G \qquad (2-17)$$

Three factors are derived for arithmetic gradients: The *P*/*G* factor for present worth, the *A*/*G* factor for annual series, and the *F*/*G* factor for future worth. There are several ways to derive them. We use the single-payment present worth factor (P/*F*, *i*, *n*), but the same result can be obtained by using the *F*/*P* , *F*/*A* , or *P*/*A* factor.

In Figure (2–6), the present worth at year 0 of only the gradient is equal to the sum of the present worth's of the individual cash flows, where each value is considered a future amount.

$$\boldsymbol{P = G(P/F,i,2) + 2G(P/F,i,3) + 3G(P/F,i,4) + \cdots + [(n-2)G]\big(P/F,i,(n-1)\big) + [(n-1)G](P/F,i,n)}$$

Factor out *G* and use the *P*/*F* formula.

$$P = G\left[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \frac{3}{(1+i)^4} + \cdots + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n}\right] \qquad (2-18)$$

Multiplying both sides of Equation (2-18) by $(1+i)^1$ yields

$$P(1+i)^1 = G\left[\frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \frac{3}{(1+i)^3} + \cdots + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}}\right] \quad (2-19)$$

Subtract Equation (2-18) from Equation (2-19) and simplify.

$$iP = G\left[\frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right] - G\left[\frac{n}{(1+i)^n}\right] (2-20)$$

The left bracketed expression is the same as that contained in Equation (2-6), where the *P*/*A* factor was derived. Substitute the closed-end form of the *P*/*A* factor from Equation (2-7).

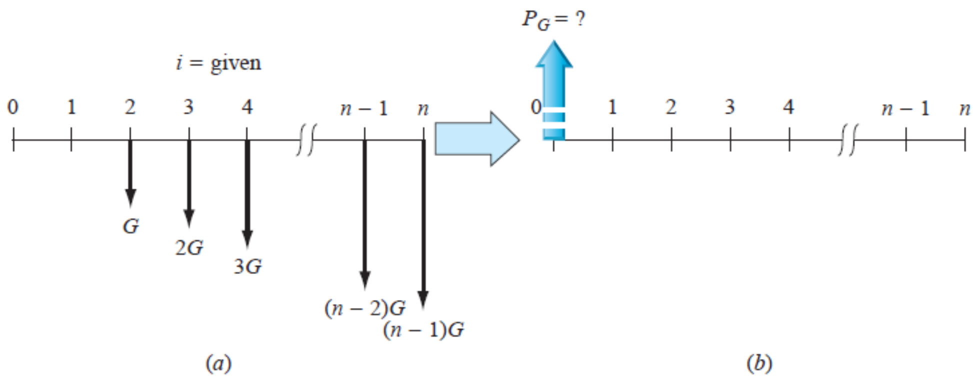


Figure (2–8): Conversion diagram from an arithmetic gradient to a present worth.

into Equation (2-20) and simplify to solve for P_G, the present worth of the gradient series only.

$$\boldsymbol{P_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]} \qquad (2-21)$$

Equation (2-21) is the general relation to **convert an arithmetic gradient *G* (not including the base amount) for *n* years into a present worth at year 0.** Figure (2–8 *a*) is converted into the equivalent cash flow in Figure (2–8 *b*). The *arithmetic gradient present worth factor*, or **P/G factor,** may be expressed in two forms:

$$\boldsymbol{(P/G,i,n) = \frac{1}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right] \; or \; (P/G,i,n) = \frac{(1+i)^n - in - 1}{i^2(1+i)^n}} \quad (2-22)$$

Remember: The conventional arithmetic gradient starts in year 2, and P is located in year 0.

Equation (2-21) expressed as an engineering economy relation is

$$P_G = G(\mathrm{P/G}, i, n) \qquad (2-23)$$

which is the rightmost term in Equation (2-16) to calculate total present worth. The *G* carries a minus sign for decreasing gradients.

The equivalent uniform annual series A_G for an arithmetic gradient *G* is found by multiplying the present worth in Equation (2-23) by the (A/*P, i, n)* formula. In standard notation form, the equivalent of algebraic cancellation of *P* can be used.

$$A_G = G(P/G, i, n)(A/P, i, n) = G(A/G, i, n)$$

In equation form,

$$A_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

$$A_G = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] \qquad (2-24)$$

which is the rightmost term in Equation (2-17). The expression in brackets in Equation (2-24) is called the *arithmetic gradient uniform series factor* and is identified by ***(A/ G, i, n)***. This factor converts Figure (2–9 a) into Figure (2–9 b).

The *P*/*G* and *A*/*G* factors and relations are summarized inside the front cover. Factor values are tabulated in the two rightmost columns of factor values at the rear of this text.

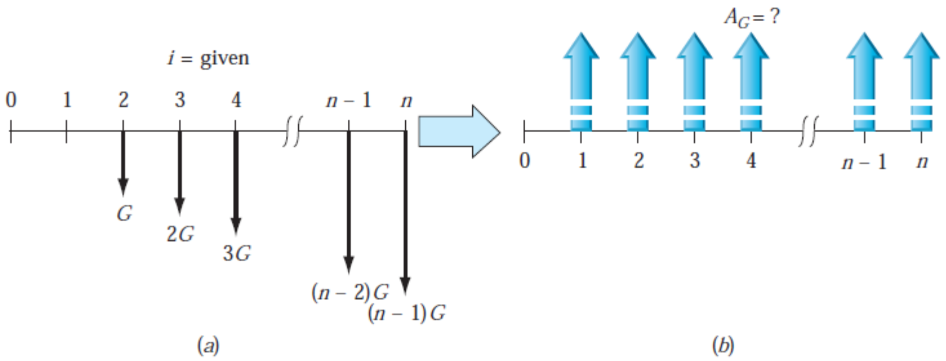


Figure (2–9): Conversion diagram of an arithmetic gradient series to an equivalent uniform annual series.

There is no direct, single-cell spreadsheet function to calculate P_G or A_G for an arithmetic gradient. Use the NPV function to display P_G and the PMT function to display A_G after entering all cash flows (base and gradient amounts) into contiguous cells. General formats for these functions are:

$$= NPV(i\%, second_{cell}: last_{cell}) + first_{cell} \qquad (2-25)$$

$$= PMT(i\%, n, cell\_with\_P_G) \qquad (2-26)$$

An ***F/G* factor** (arithmetic *gradient future worth factor)* to calculate the future worth *F G* of a gradient series can be derived by multiplying the *P/G* and *F/P* factors. The resulting factor, (F/*G*, *i*, *n)*, in brackets, and engineering economy relation is

$$\boldsymbol{F_G = G\left[\left(\frac{1}{i}\right)\left(\frac{(1+i)^n - 1}{i}\right) - n\right]}$$

Example (2-5)

Neighboring parishes in Louisiana have agreed to pool road tax resources already designated for bridge refurbishment. At a recent meeting, the engineers estimated that a total of $500,000 will be deposited at the end of next year into an account for the repair of old and safety-questionable bridges throughout the area. Further, they estimate that the deposits will increase by $100,000 per year for only 9 years thereafter, then cease.

1) Determine the equivalent present worth
2) Determine the equivalent annual series amounts, if public funds earn at a rate of 5% per year.

Solution

1) The cash flow diagram of this conventional arithmetic gradient series from the perspective of the parishes is shown in Figure (2–10). According to Equation (2-16), two computations must be made and added: the first for the present worth of the base amount P_A and the second for the present worth of the gradient P_G. The total present worth P_T occurs in year 0. This is illustrated by the partitioned cash flow diagram in Figure (2–11). In $1000 units, the total present worth is

$$\begin{aligned} P_T &= 500(P/A, 5\%, 10) + 100(P/G, 5\%, 10) \\ &= 500(7.7217) + 100(31.6520) \\ &= \$7026.05\ (\$7{,}026{,}050) \end{aligned}$$

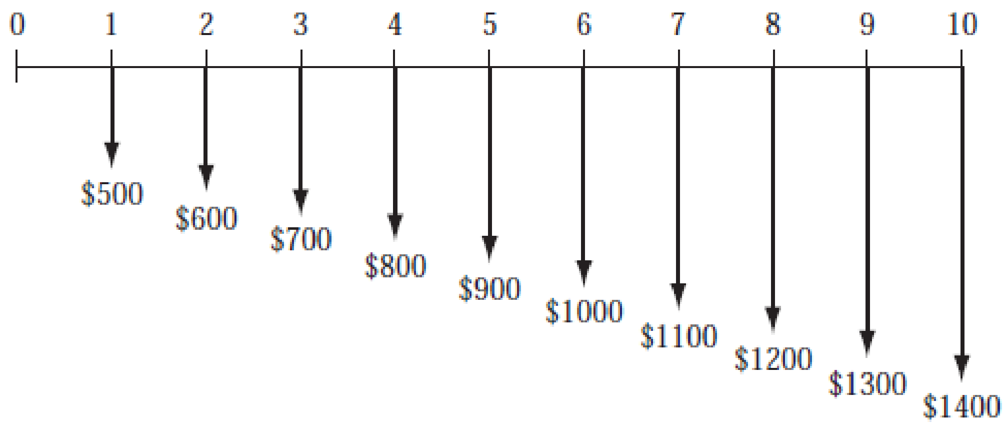


Figure (2–10): Cash flow series with a conventional arithmetic gradient (in $1000 units), Example (2-5).

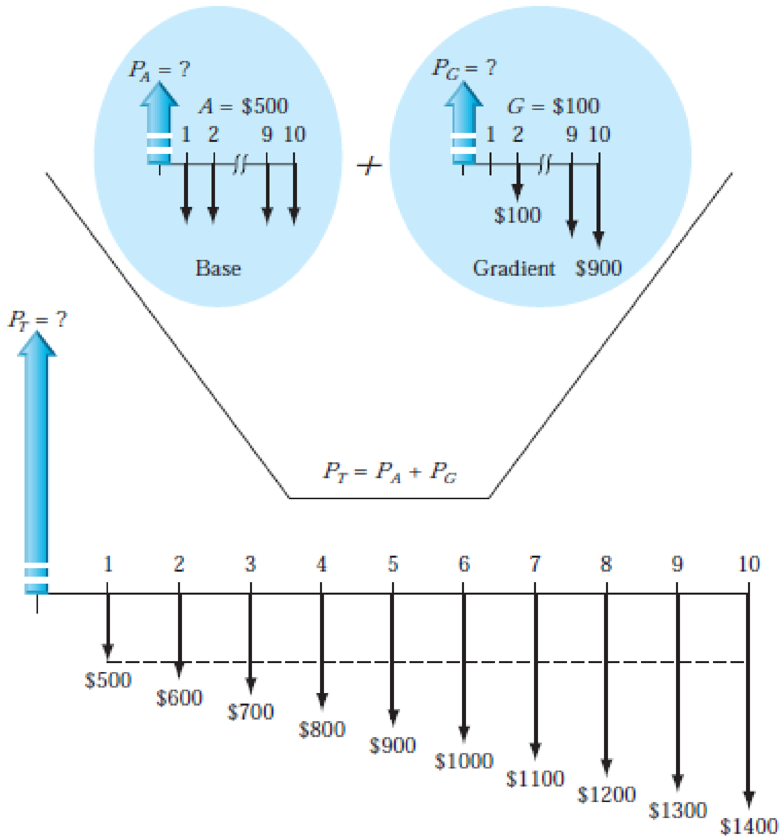


Figure (2–11): Partitioned cash flow diagram (in $1000 units), Example (2-5).

2) Here, too, it is necessary to consider the gradient and the base amount separately. The total annual series A_T is found by Equation (2-17) and occurs in years 1 through 10.

$$A_T = 500 + 100(A/G, 5\%, 10) = 500 + 100(4.0991)$$
$$= \$909.91 \text{ per year} \quad (\$909.910)$$

4. Geometric Gradient Series Factors

It is common for annual revenues and annual costs such as maintenance, operations, and labor to go up or down by a constant percentage, for example, +5% or -3% per year. This change occurs every year on top of a starting amount in the first year of the project. A definition and description of new terms follow.

A geometric gradient series is a cash flow series that either increases or decreases by a constant percentage each period. The uniform change is called the rate of change.

- $\boldsymbol{g}$ = constant rate of change, in decimal form, by which cash flow values increase or decrease from one period to the next. The gradient g can be + or -.
- $\boldsymbol{A_1}$= initial cash flow in year 1 of the geometric series
- $\boldsymbol{P_g}$ = present worth of the entire geometric gradient series, including the initial amount A_1

Note that the initial cash flow A_1 is not considered separately when working with geometric gradients. Figure (2–12) shows increasing and decreasing geometric gradients starting at an amount A_1 in time period 1 with present worth P_g located at time 0. The relation to determine the total present worth P_g for the entire cash flow series may be derived by multiplying each cash flow in Figure (2–12 a) by the *P/F* factor $1/(1+i)^n$.

$$P_g = \frac{A_1}{(1+i)^1} + \frac{A_1(1+g)}{(1+i)^2} + \frac{A_1(1+g)^2}{(1+i)^3} + \cdots + \frac{A_1(1+g)^{n-1}}{(1+i)^n}$$

$$= A_1\left[\frac{1}{(1+i)^1} + \frac{(1+g)}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \cdots + \frac{(1+g)^{n-1}}{(1+i)^n}\right] \quad (2-27)$$

Multiply both sides by $(1+g)/(1+i)$, subtract Equation (2-27) from the result, factor out P_g, and obtain

$$P_g\left(\frac{(1+g)}{(1+i)} - 1\right) = A_1\left[\frac{(1+g)^n}{(1+i)^{n+1}} - \frac{1}{1+i}\right]$$

Solve for P_g and simplify.

$$P_g = A_1\left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right] \; ; \quad g \neq i \qquad (2-28)$$

The term in brackets in Equation (2-27) is the **(*P/A, g, i, n*)** or *geometric gradient series present worth factor* for values of g not equal to the interest rate i. When $g = i$, substitute i for g in Equation (2-28) and observe that the term $1/(1+i)$ appears n times.

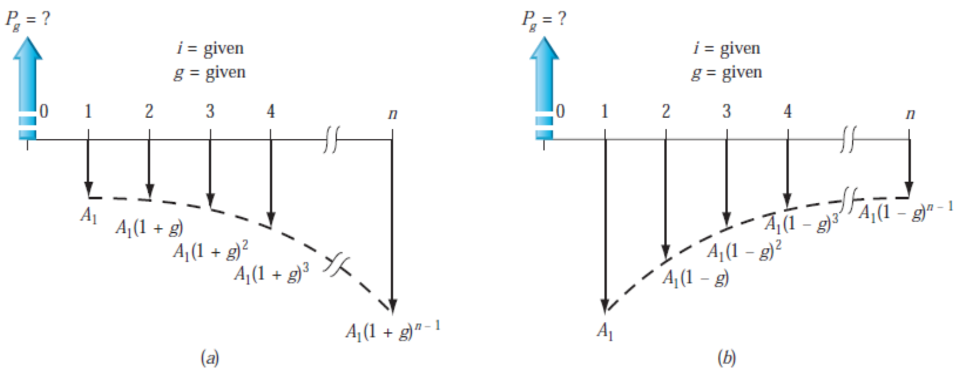


Figure (2–12): Cash flow diagram of (a) increasing and (b) decreasing geometric gradient series and present worth P_g.

$$P_g = A_1\left[\frac{1}{(1+i)} + \frac{1}{(1+i)} + \frac{1}{(1+i)} + \cdots + \frac{1}{(1+i)}\right]$$

$$P_g = \frac{nA_1}{(1+i)} \qquad (2-29)$$

The (P/A, g, i, n) factor calculates P_g in period t = 0 for a geometric gradient series starting in period 1 in the amount A_1 and increasing by a constant rate of g each period.

The equation for P_g and the $(P/A, g, i, n)$ factor formula are

$$P_g = A_1(P/A, g, i, n) \qquad (2-30)$$

$$(P/A, g, i, n) = \begin{cases} \dfrac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} & ; \; g \neq i \\ \dfrac{n}{(1+i)} & ; \; g = i \end{cases} \qquad (2-31)$$

It is possible to derive factors for the equivalent A and F values; however, it is easier to determine the P_g amount and then multiply by the A/P or F/P factor. As with the arithmetic gradient series, there are no direct spreadsheet functions for geometric gradient series. Once the cash flows are entered, P and A are determined using the NPV and PMT functions, respectively.

Example (2-6)

A coal-fi red power plant has upgraded an emission control valve. The modification costs only \$8000 and is expected to last 6 years with a \$200 salvage value. The maintenance cost is expected to be high at \$1700 the first year, increasing by 11% per year thereafter. Determine the equivalent present worth of the modification and maintenance cost by hand and by spreadsheet at 8% per year.

Solution

1) Solution by Hand

The cash flow diagram Figure (2–13) shows the salvage value as a positive cash flow and all costs as negative. Use Equation (2-35) for $g \neq i$ to calculate P_g. Total P_T is the sum of three present worth components.

$$P_T = -8000 - P_g + 200(P/F, 8\%, 6)$$

$$= -8000 - \left[\frac{1-\left(\frac{1.11}{1.08}\right)^6}{0.08-0.11}\right] + 200(P/F, 8\%, 6)$$

$$= -8000 - 1700(5.9559) + 120 = \$-17{,}999$$

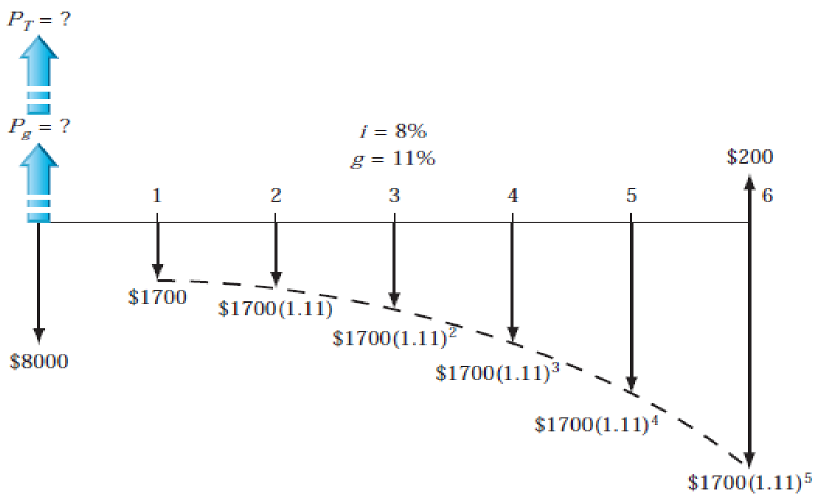
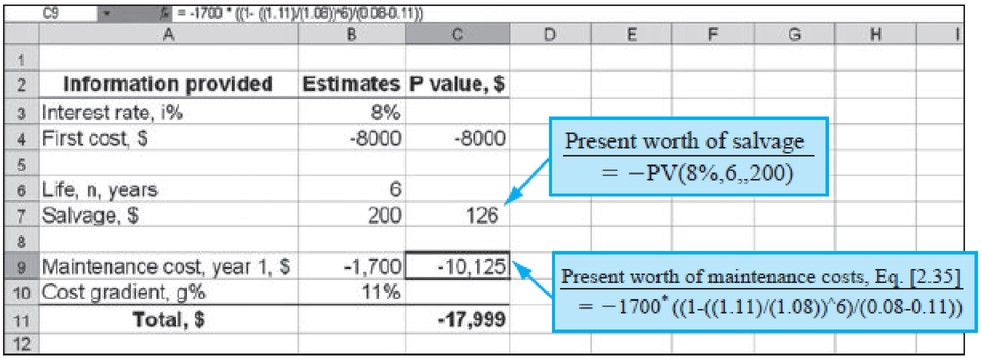


Figure (2–13): Cash flow diagram of a geometric gradient, Example (2-6).

2) Solution by Spreadsheet

Figure (2–14) details the spreadsheet operations to find the geometric gradient present worth P_g and total present worth P_T. To obtain P_T = $-17,999, three components are summed—first cost, present worth of estimated salvage in year 6, and P_g. Cell tags detail the relations for the second and third components; the first cost occurs at time 0.

The relation that calculates the (P/A, g, $i\%$, n) factor is rather complex, as shown in the cell tag and formula bar for C9. If this factor is used repeatedly, it is worthwhile using cell reference formatting so that A_1, i, g, and n values can be changed and the correct value is always obtained. Try to write the relation for cell C9 in this format.


C9 = -1700 * ((1- ((1.11)/(1.08))^6)/(0.08-0.11))

	A	B	C
1			
2	Information provided	Estimates	P value, $
3	Interest rate, i%	8%	
4	First cost, $	-8000	-8000
5			
6	Life, n, years	6	
7	Salvage, $	200	126
8			
9	Maintenance cost, year 1, $	-1,700	-10,125
10	Cost gradient, g%	11%	
11	Total, $		-17,999
12			

Figure (2–14): Geometric gradient and total present worth calculated via spreadsheet, Example (2-6).

5. Calculations for Cash Flows That Are Shifted

When a uniform series begins at a time other than at the end of period 1, it is called a shifted series. In this case several methods based on factor equations or tabulated values can be used to find the equivalent present worth P.

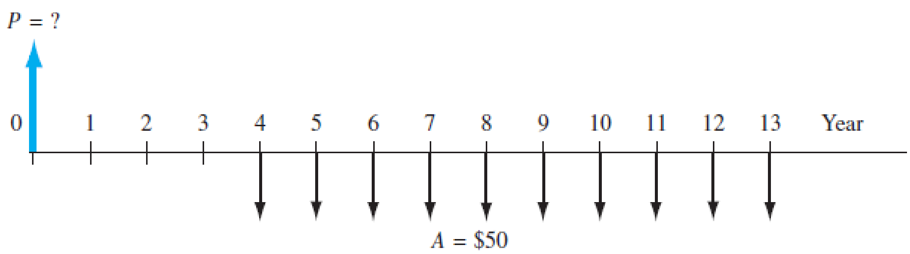


Figure (2–15): A uniform series that is shifted.

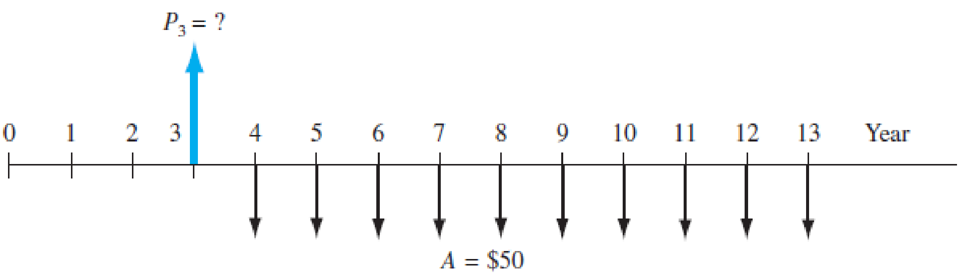


Figure (2–16): Location of present worth for the shifted uniform series in Figure (2–15)

For example, P of the uniform series shown in Figure (2-15) could be determined by any of the following methods:

- Use the *P/F* factor to find the present worth of each disbursement at year 0 and add them.
- Use the *F/P* factor to find the future worth of each disbursement in year 13, add them, and then find the present worth of the total using *P=F (P/F, i, 13).*
- Use the *F/A* factor to find the future amount *F=A (F/A, i, 10)*, and then compute the present worth using *P = F (P/F, i, 3).*
- Use the *P/A* factor to compute the "present worth" (which will be located in year 3 not year 0), and then find the present worth in year 0 by using the *(P/F, i, 3)* factor. (Present worth is enclosed in quotation marks here only to represent the present worth as determined by the *P/A* factor in year 3, and to differentiate it from the present worth in year 0.)

Typically, the last method is used. For Figure (2-15), the "present worth" obtained using the *P/A* factor is located in year 3. This is shown as P_3 in Figure (2-16).

Remember, the present worth is always located one period prior to the first uniform-series amount when using the P/A factor.

To determine a future worth or *F* value, recall that the *F/A* factor has the *F* located in the *same* period as the last uniform-series amount. Figure (2-17) shows the location of the future worth when *F/A* is used for Figure (2-15) cash flows.

Remember, the future worth is always located in the same period as the last uniform-series amount when using the F/A factor.

It is also important to remember that the number of periods *n* in the *P/A* or *F/A* factor is equal to the number of uniform-series values. It may be helpful to *renumber* the cash flow diagram to avoid errors in counting. Figure (2-17) shows Figure (2-15) renumbered to determine *n* = 10.

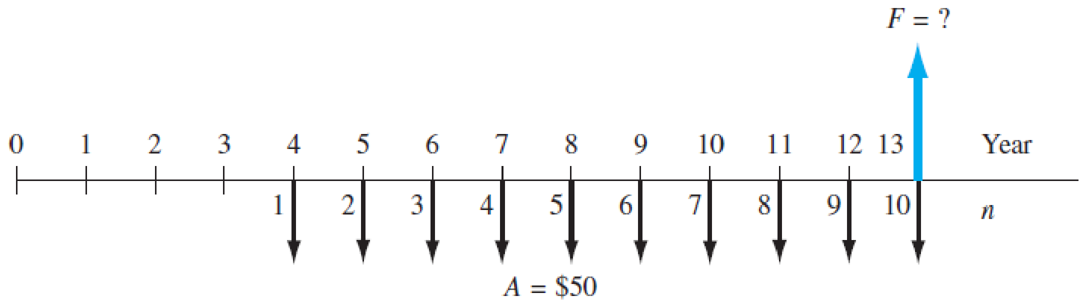


Figure (2–17): Placement of F and renumbering for n for the shifted uniform series of Figure (2-15)

As stated above, there are several methods that can be used to solve problems containing a uniform series that is shifted. However, it is generally more convenient to use the uniform-series factors than the single-amount factors. There are specific steps that should be followed in order to avoid errors:

1. Draw a diagram of the positive and negative cash flows.
2. Locate the present worth or future worth of each series on the cash flow diagram.
3. Determine *n* for each series by renumbering the cash flow diagram.
4. Set up and solve the equations.

Example (2-7)

An engineering technology group just purchased new CAD software for $5000 now and annual payments of $500 per year for 6 years starting 3 years from now for annual upgrades. What is the present worth of the payments if the interest rate is 8% per year?

Solution

The cash flow diagram is shown in Figure (2-18). The symbol P_A is used to represent the present worth of a uniform annual series A, and P_A' represents the present worth at a time other than period 0. Similarly, P_T represents the total present worth at time 0. The correct placement of P_A' and the diagram renumbering to obtain n are also indicated.

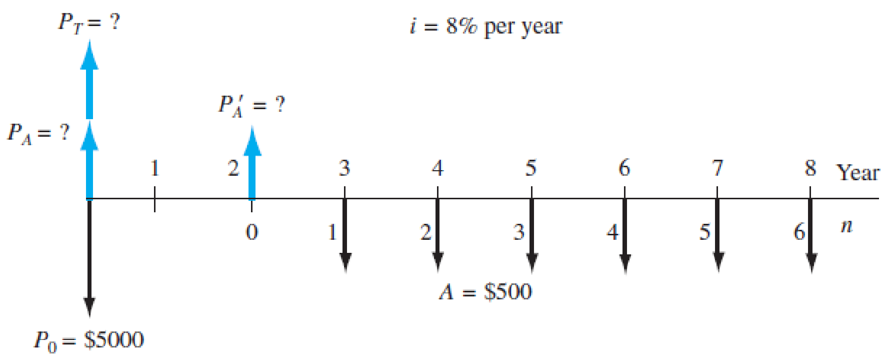


Figure (2–18): Cash flow diagram with placement of *P* values, Example (2-6)

Note that P_A' is located in actual year 2, not year 3. Also, $n = 6$ not 8, for the P/A factor. First find the value of P_A' the shifted series.

$$P_A' = \$500(P/A, 8\%, 6)$$

Since P_A' is located in year 2, now find P_A in year 0.

$$P_A = P_A'(P/F, 8\%, 2)$$

The total present worth is determined by adding P_A and the initial payment P_0 in year 0.

$$\begin{aligned} P_T &= P_0 + P_A \\ &= 5000 + 500(P/A, 8\%, 6)(P/F, 8\%, 2) \\ &= 5000 + 500(4.6229)(0.8573) \\ &= \$6981.60 \end{aligned}$$

To determine the present worth for a cash flow that includes both uniform series and single amounts at specific times, use the P/F factor for the single amounts and the P/A factor for the series. To calculate A for the cash flows, first convert everything to a P value in year 0, or an F value in the last year. Then obtain the A value using the A/P or A/F factor, where n is the total number of years over which the A is desired.

Many of the considerations that apply to shifted uniform series apply to arithmetic gradient series as well. Recall that a conventional gradient series starts between periods 1 and 2 of the cash flow sequence. A gradient starting at any other time is called a *shifted gradient.* The n value in the P/G and A/G factors for the shifted gradient is determined by renumbering the time scale. The period in which the *gradient first appears is labeled period 2.* The n value for the factor is determined by the renumbered period where the last gradient increase occurs. The P/G

factor values and placement of the gradient series present worth P_G for the shifted arithmetic gradients in Figure (2-19) are indicated.

It is important to note that the A/G factor *cannot* be used to find an equivalent A value in periods 1 through n for cash flows involving a shifted gradient. Consider the cash flow diagram of Figure (2-19 b). To find the equivalent annual series in years 1 through 10 for the arithmetic gradient series only, first find the present worth of the gradient in year 5, take this present worth back to year 0, and then annualize the present worth for 10 years with the A/P factor. If you apply the annual series gradient factor (A/G, i, 5) directly, the gradient is converted into an equivalent annual series over years 6 through 10 only.

Remember, to find the equivalent A series of a shifted gradient through all of the periods, first find the present worth of the gradient at actual time 0, then apply the (A/P, i, n) factor.

If the cash flow series involves a *geometric gradient* and the gradient starts at a time other than between periods 1 and 2, it is a shifted gradient. The P_g is located in a manner similar to that for P_G above, and Equation (2-28) is the factor formula.

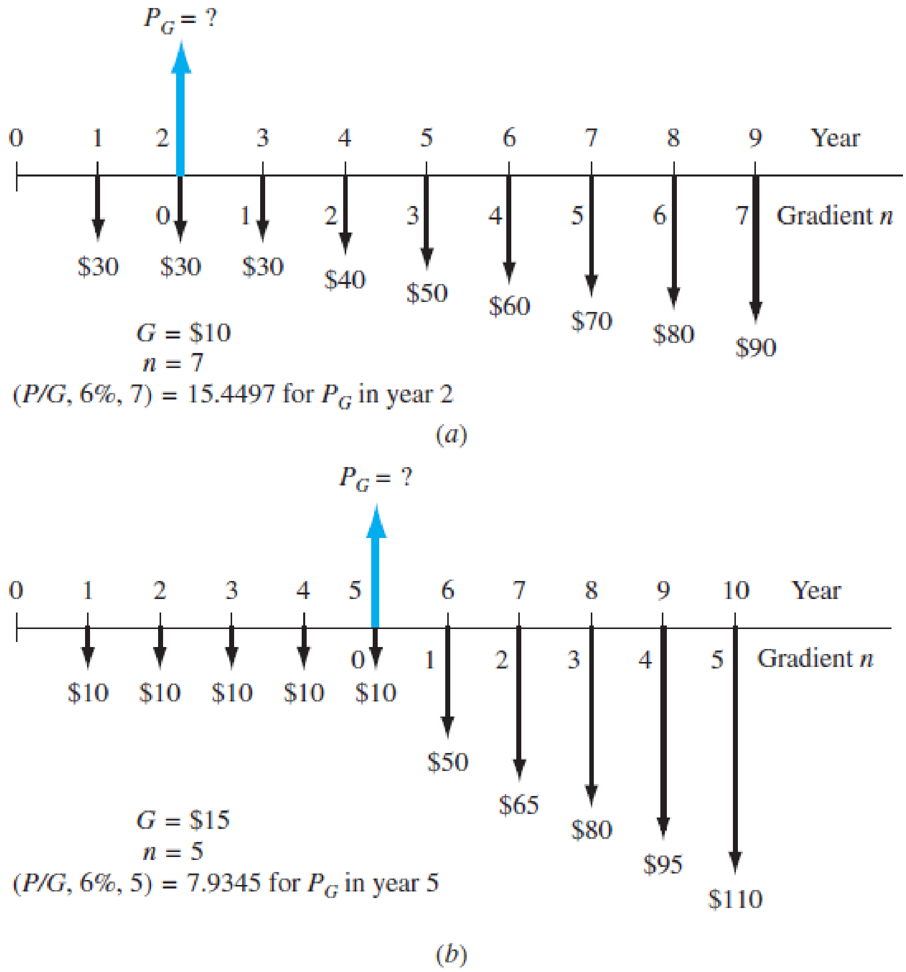


Figure (2–19): Determination of G and n values used in factors for shifted gradients.

Example (2-8)

Chemical engineers at a Coleman Industries plant in the Midwest have determined that a small amount of a newly available chemical additive will increase the water repellency of Coleman's tent fabric by 20%. The plant superintendent has arranged to purchase the additive through a 5-year contract at $7000 per year, starting 1 year from now. He expects the annual price to increase by 12% per year starting in the sixth year and thereafter through year 13. Additionally, an initial investment of $35,000 was made now to prepare a site suitable for the contractor to deliver the additive. Use *i = 15%* per year to determine the equivalent total present worth for all these cash flows.

Solution

Figure (2-20) presents the cash flows. The total present worth PT is found using $g = 0.12$ and $i = 0.15$. Equation (2-28) is used to determine the present worth P_g for the entire geometric series at actual year 4, which is moved to year 0 using $(P/F, 15\%, 4)$.

$$P_T = 35{,}000 + A(P/A, 15\%, 4) + A_1(P/A, 12\%, 15\%, 9)(P/F, 15\%, 4)$$

$$= 35{,}000 + 7000(2.8550) + \left[7000 \frac{1 - \left(\frac{1.12}{1.15}\right)^9}{0.15 - 0.12}\right](0.5718)$$

$$= 35{,}000 + 19{,}985 + 28{,}247$$

$$= \$83{,}232$$

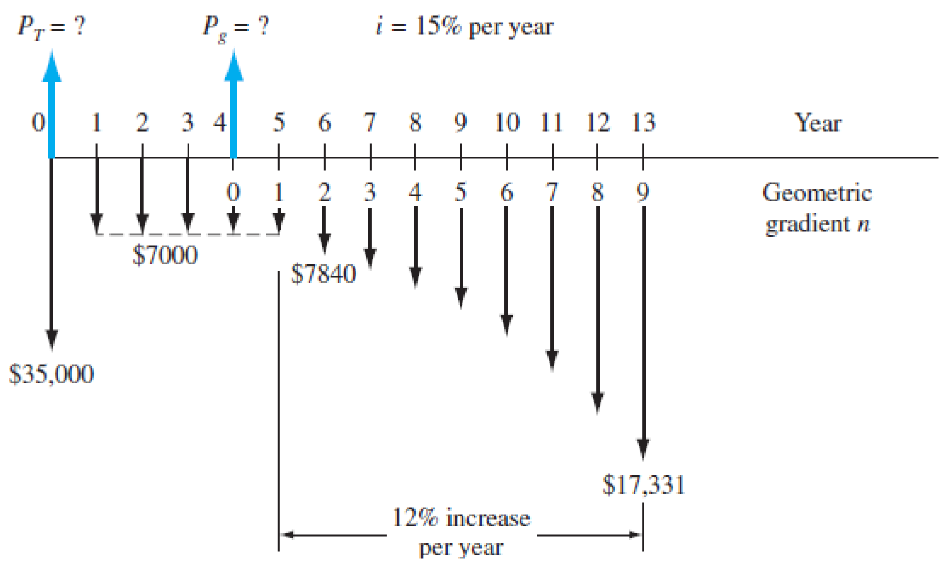


Figure (2–20): Cash flow diagram including a geometric gradient with g = 12%, Example (2-8).

Note that $n = 4$ in the $(P/A,15\%,4)$ factor because the $7000 in year 5 is the initial amount A_1 in Equation (2-20) for the geometric gradient.

6. Using Spreadsheets for Equivalency Computation

The easiest single-cell spreadsheet functions to apply to find *P*, *F*, or *A* require that the cash flows exactly fit the function format. The functions apply the correct sign to the answer that would be on the cash flow diagram. That is, if cash flows are deposits (minus), the answer will have a plus sign. In order to retain the sign of the inputs, enter a minus sign prior to the function. Here is a summary and examples at 5% per year.

- **Present worth *P*:** Use the *PV* function = *(i%, n, A, F)* if *A* is exactly the same for each of *n* years; *F* can be present or not. For example, if *A* = $3000 per year deposit for *n = 10* years, the function = *PV (5%, 10, -3000)* will display *P = $23,165*. This is the same as using the *P/A* factor to find *P = 3000(P/A,5%,10) = 3000(7.7217) = $23,165.*
- **Future worth *F*:** Use the FV function = *FV (i%, n, A, P)* if *A* is exactly the same for each of *n* years; *P* can be present or not. For example, if *A* = $3000 per year deposit for *n* = 10 years, the function = *FV (5%, 10, -3000)* will display *F* = $37,734. This is the same as using the *F/A* factor to find *F = 3000(F/A, 5%, 10) = 3000(12.5779) = $37,734.*
- **Annual amount *A*:** Use the *PMT* function = *PMT(i%, n, P, F)* when there is no *A* present, and either *P* or *F* or both are present. For example, for *P*= $-3000 deposit now and *F* = $5000 returned *n = 10* years hence, the function =*-PMT (5%, 10, -3000, 5000)* will display *A*= $9. This is the same as using the *A/P* and *A/F* factors to find the equivalent net *A*= $9 per year between the deposit now and return 10 years later.

$$A = -3000\ (A/P,5\%,10) + 5000\ (A/F,5\%,10) = -389 + 398 = \$9$$

- **Number of periods *n*:** Use the *NPER* function = *NPER(i%, A, P, F)* if *A* is exactly the same for each of *n* years; either *P* or *F* can be omitted, but not both. For example, for *P* = $-25,000 deposit now and *A* = $3000 per year return, the function = *NPER(5%,3000, -25000)* will display *n* = 11.05 years to recover *P* at 5% per year. This is the same as using trial and error to find *n* in the relation *0 = -25,000 + 3,000 (P/A,5%, n).*

When cash flows vary in amount or timing, it is usually necessary to enter them on a spreadsheet, including all zero amounts, and utilize other functions for P, F, or A values. All spreadsheet functions allow another function to be embedded in them, thus reducing the time necessary to get final answers. Example (2-9) illustrates these functions and the embedding capability. Example (2-10) demonstrates how easily spreadsheets handle arithmetic and percentage gradients and how the IRR (rate of return) function works.

Example (2-9)

Carol just entered college and her grandparents have offered her one of two gifts. They promised to give her $25,000 toward a new car if she graduates in 4 years. Alternatively, if she takes 5 years to graduate, they offered her $5000 each year starting after her second year is complete and an extra $5000 when she graduates. Draw the cash flow diagrams first. Then, use *i = 8%* per year to show Carol how to use spreadsheet functions and her financial calculator TVM functions to determine the following for *each gift* offered by her grandparents.

1) Present worth P now
2) Future worth F five years from now
3) Equivalent annual amount A over a total of 5 years
4) Number of years it would take Carol to have $25,000 in hand for the new

car if she were able to save $5000 each year starting next year.

Solution

Spreadsheet: The two cash flow series, labeled Gift A (lump sum) and Gift B (spread out), are in Figure (2-21). The spreadsheet in Figure (2-22a) lists the cash flows (don't forget to enter the $0 cash flows so the NPV function can be used), and answers to each part using the PV, NPV, FV, or PMT functions as explained below. In some cases, there are alternative ways to obtain the answer. Figure (2-22b) shows the function formulas with some comments. Remember that the PV, FV, and PMT functions will return an answer with the opposite sign from that of the cash flow entries. The same sign is maintained by entering a minus before the function name.

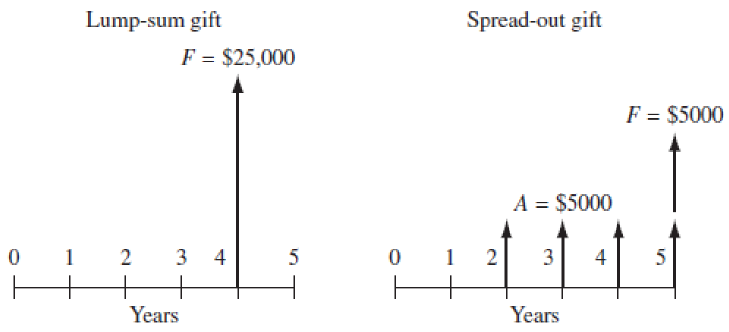


Figure (2–21): Cash flows for Carol's gift from her grandparents, Example (2-9).

1) Rows 12 and 13: There are two ways to find *P*; either the PV or NPV function. NPV requires that the zeros be entered. (For Gift A, omitting zeros in years 1, 2, and 3 will give the incorrect answer of *P*= $23,148, because NPV assumes the $25,000 occurs in year 1 and discounts it only one year at 8%.) The single-cell PV is hard to use for Gift B since cash flows do not start until year 2; using NPV is easier.
2) Rows 16 and 17: There are two ways to use the FV function to find *F* at the end of year 5. To develop FV correctly for Gift B in a single cell without listing cash flows, add the extra $5000 in year 5 separate from the FV for the four *A* = $5000 values. Alternatively, cell D17 embeds the NPV function for the *P* value into the FV function. This is a very convenient way to combine functions.
3) Rows 20 and 21: There are two ways to use the PMT function to find *A* for 5 years; find *P* separately and use a cell reference or embed the NPV function into the PMT to find *A* in one operation.
4) Row 24: Finding the years to accumulate $25,000 by depositing $5000 each year using the NPER function is independent of either plan. The entry *= NPER (8%, -5000,25000)* results in 4.3719 years. This can be confirmed by calculating *5000(F/A,8%,4.3719) = 5000(5.0000) = $25,000* (The 4.37 years is about the time it will take Carol to finish college. Of course, this assumes she can actually save $5000 a year while working on the degree.)

Calculator: Table (2-3) shows the format and completed calculator function for each gift, followed by the numerical answer below it. Minus signs on final answers have been changed to plus as needed to reflect the same sense as that in the spreadsheet solution. When calculating the values for Gift B, the functions can be performed separately, as shown, or embedded in the same way as the spreadsheet functions are embedded in Figure (2-22). In all cases, the answers are identical for the spreadsheet and calculator solutions.

	A	B	C	D
1			Cash flow, $	
2		Year	Gift A	Gift B
3		0		
4		1	0	0
5		2	0	5,000
6		3	0	5,000
7	Be sure to enter each zero cash flow	4	25,000	5,000
8		5	0	10,000
9				
10				
11		Function applied		
12	a. Present worth now	PV (single cell)	$18,376	
13		NPV		$18,737
14				
15				
16	b. Future worth; year 5	FV (single cell)	$27,000	$27,531
17		FV with embedded NPV		$27,531
18				
19				
20	c. Annual worth; years 1–5	PMT (reference P)	$4,602	$4,693
21		PMT with embedded NPV	$4,602	$4,693
22				
23				
24	d. Years to $25,000	NPER for both gifts	4.37	4.37
25				
26				

(*a*)

	A	B	C	D	E
1			Cash flow, $		
2		Year	Gift A	Gift B	
3		0			
4		1	0	0	
5		2	0	5000	
6		3	0	5000	
7		4	25000	5000	
8		5	0	10000	
9					
10					
11		Function applied			
12	a. Present worth now	PV with n = 4	= -PV(8%,4,,25000)		
13		NPV	= NPV(8%,C4:C8)	= NPV(8%,D4:D8)	
14					
15					
16	b. Future worth; year 5	FV	= -FV(8%,1,,25000)	= -FV(8%,4,5000) + 5000	
17		FV with embedded NPV		= -FV(8%,5,,NPV(8%,D4:D8))	
18					
19					
20	c. Annual worth; years 1–5	PMT (reference P)	= -PMT(8%,5,C12)	= -PMT(8%,5,D13)	
21		PMT with embedded NPV	= -PMT(8%,5,NPV(8%,C4:C8))	= -PMT(8%,5,NPV(8%,D4:D8))	
22					
23					
24	d. Years to $25,000	NPER (same for both)	= NPER(8%,-5000,,25000)	= NPER(8%,-5000,,25000)	
25					
26					

Function recognizes extra $5000 in year 5 as F value; no cash flow listing needed

Embedded NPV finds P in year 0

Functions reference P determined using PV or NPV function

(*b*)

Figure (2–22): (a) Use of several spreadsheet functions to find P, F, A, and n values, and (b) format of functions to obtain values, Example (2-9).

Table (2–3): Solution Using Calculator TVM Functions, Example (2-9)

Year	Cash flow, $	
	Gift A	Gift B
0		
1	0	0
2	0	5,000
3	0	5,000
4	25,000	5,000
5	0	5,000 + 5,000
	Functions applied	
a. Present worth now	PV(i,n,A,F) PV(8,4,0,25000) **$18,376**	FV(i,n,A,P) + 5,000 FV(8,4,5000,0) + 5,000 **$27,531** PV($i$,$n$,$A$,$F$) PV(8,5,0,27531) **$18,737**
b. Future worth, year 5	FV(i,n,A,P) FV(8,1,0,25000) **$27,000**	FV(i,n,A,P) + 5,000 FV(8,4,5000,0) + 5,000 **$27,531**
c. Annual worth, years 1–5	PMT(i,n,P,F) PMT(8,5,0,27000) **$4,602**	PMT(i,n,P,F) PMT(8,5,0,27531) **$4,693**
d. Years to $25,000	n(i,A,P,F) n(8,−5000,0,25000) **4.37**	n(i,A,P,F) n(8,−5000,0,25000) **4.37**

Example (2-10)

Bobby was desperate. He borrowed $600 from a pawn shop and understood he was to repay the loan starting next month with $100, increasing by $10 per month for a total of 8 months. Actually, he misunderstood. The repayments increased by 10% each month after starting next month at $100. Use a spreadsheet to calculate the *monthly* interest rate that he thought he was to pay, and what he actually will pay.

Solution

Figure (2-23) lists the cash flows for the assumed arithmetic gradient G = $10 per month, and the actual percentage gradient g = 10% per month. Note the simple relations to construct the increasing cash flows for each type gradient. Apply the IRR function to each series using its format = IRR (first_cell: last_cell). Bobby is paying an exorbitant rate per month (and year) at 14.9% per month, which is higher than he expected it to be at 13.8% per month.

If needed to solve a problem, the tables in the rear of this text provide the numerical value for any of the six common compound interest factors. However, the desired i or n may not be tabulated. Then the factor formula can be applied to obtain the numerical value; plus, a spreadsheet or calculator function can be used with a “1” placed in the P, A, or F location in the function. The other parameter is omitted or set to “0.”

For example, the *P*/*F* factor is determined using the spreadsheet's PV function with the *A* omitted (or set to 0) and *F* = 1, that is, = -PV(*i*,*n*,,1) or = -PV(*i*,*n*,0,1). The minus sign makes the result positive. If a calculator is used, the functional notation is PV(*i*,*n*,0,1) for the function PV(*i*,*n*,*A*,*F*). Table (2-4) summarizes the notation for spreadsheets and calculators. This information, in abbreviated form, is included inside the front cover.

	A	B	C	D	E	F	G	H
1		Cash flow, $			Cash flow, $			
2	Month	G = $10			g = 10%			
3	0	600.00			600.00			
4	1	-100.00	= B4-10		-100.00	= E4*(1.1)		
5	2	-110.00			-110.00			
6	3	-120.00			-121.00			
7	4	-130.00			-133.10			
8	5	-140.00			-146.41			
9	6	-150.00			-161.05			
10	7	-160.00			-177.16			
11	8	-170.00			-194.87	= SUM(E4:E11)		
12	Total paid back	-1080.00			-1143.59			
13	ROR per month	**13.8%**	= IRR(B3:B11)		**14.9%**	= IRR(E3:E11)		
14								
15								

Figure (2–23): Use of a spreadsheet to generate arithmetic and percentage gradient cash flows and application of the IRR function, Example (2-10).

When using a spreadsheet, an unknown value in one cell may be required to force the value in a different cell to equal a stated value. For example, the present worth of a given cash flow series is known to equal $10,000 and all but one of the cash flow values is known. This unknown cash flow is to be determined. The spreadsheet tool called GOAL SEEK is easily applied to find one unknown value.

If the Factor is:	To Do This:	The Spreadsheet Function is:	The Calculator Function is:
A/P	*Find P, given F*	*= -PV(i,n,,1)*	*PV(i,n,0,1) for PV(i,n,A,F)*
F/P	*Find F, given P*	*= -FV(i,n,,1)*	*FV(i,n,0,1) for FV(i,n,A,P)*
A/F	*Find A, given F*	*= -PMT(i,n,,1)*	*PMT(i,n,0,1) for PMT(i,n,P,F)*
F/A	*Find F, given A*	*= -FV(i,n,1)*	*FV(i,n,1,0) for FV(i,n,A,P)*
P/A	*Find P, given A*	*= -PV(i,n,1)*	*PV(i,n,1,0) for PV(i,n,A,F)*
A/P	*Find A, given P*	*= -PMT(i,n,1)*	*PMT(i,n,1,0) for PMT(i,n,P,F)*

Problems

(2-1) Find the correct numerical value for the following factors from the interest tables:
a) (F/P,10%,20)
b) (A/F,4%,8)
c) (P/A,8%,20)
d) (A/P,20%,28)
e) (F/A,30%,15)

(2-2) What is the present worth of $30,000 in year 8 at an interest rate of 10% per year?

(2-3) The Moller Skycar M400 is a flying car known as a personal air vehicle (PAV). The cost is $995,000, and a $100,000 deposit holds one of the first 100 vehicles. Assume a buyer pays the $885,000 balance 3 years after making the $100,000 deposit. At an interest rate of 10% per year, determine the effective total cost of the PAV in year 3 using:
a) Tabulated factors,
b) A single-cell spreadsheet function.

(2-4) How much will be in an investment account 12 years from now if you deposit $3000 now and $5000 four years from now and the account earns interest at a rate of 10% per year? Use:
a) Tabulated factor values,
b) TVM functions on a financial calculator,
c) Built-in functions on a spreadsheet.

(2-5) Ametek Technical & Industrial Products (ATIP) manufactures brushless blowers for boilers, foodservice equipment, and fuel cells. The company borrowed $17,000,000 for a plant expansion and repaid the loan in eight annual payments of $2,737,680, with the first payment made one year after the company received the money. What interest rate did ATIP pay? Develop the answer using:
a) Tabulated factor values,
b) A financial calculator,
c) Spreadsheet functions.

(2-6) How many years will it take for money to increase to three times the initial amount at an interest rate of 10% per year?

(2-7) Acceleron is planning future expansion with a new facility in Indianapolis. The company will make the move when its real estate sinking fund has a total value of $1.2 million. If the fund currently has $400,000 and the company adds $50,000 per year, how many years will it take for the account to reach the desired value? The fund earns interest at a rate of 10% per year.

(2-8) Silastic-LC-50 is a liquid silicon rubber designed to provide high clarity, superior mechanical properties, and short cycle time for high speed manufacturers. One high-volume manufacturer used it to achieve smooth release from molds. The company's projected growth would result in silicon costs of $26,000 next year and costs increasing by $2000 per year through year 5. The interest rate is 10% per year.
a) What is the present worth of these costs using tabulated factors?
b) How is this problem solved using a spreadsheet? Using a financial calculator?

(2-9) For the cash flows shown, determine the value of G that makes the present worth in year 0 equal to $14,000. The interest rate is 10% per year.

Year	0	1	2	3	4
Cash flow, $ per year	—	8000	8000-G	8000-2G	8000-3G

(2-10) For the cash flow series shown, determine the future worth in year 5 at an interest rate of 10% per year.

Year	1	2	3	4	5
Cash flow, $	300,000	275,000	250,000	225,000	200,000

(2-11) Attenuated Total Reflectance (ATR) is a method for looking at the surfaces of materials that are too opaque or too thick for standard transmission methods. A manufacturer of precision plastic parts estimates that ATR spectroscopy can save the company $8000 per year by reducing returns of out-of-spec parts. What is the future worth of the savings if they start now and extend through year 4? Use i = 10% per year.

(2-12) For the cash flows shown, calculate the future worth in year 8 at i = 10% per year.

Year	0	1	2	3	4	5	6
Cash flow, $	100	100	100	200	200	200	200

(2-13) Encon Systems, Inc. sales revenues for a product line introduced 7 years ago is shown. Use tabulated factors, a calculator or a spreadsheet to calculate the equivalent annual worth over the 7 years using an interest rate of 10% per year.

Year	**Revenue, $**
0	4,000,000
1	4,000,000
2	4,000,000
3	4,000,000
4	5,000,000
5	5,000,000
6	5,000,000
7	5,000,000

Choose the correct answers for the following questions.

(2-1) A mechanical engineer conducting an economic analysis of wireless technology alternatives discovered that the F/G factor values were not in his table. He decided to create the (F/G,i,n) factor values himself. He did so by:

a) Multiplying the (F/A,i,n) and (A/G,i,n) values
b) Dividing (F/A,i,n) values by (A/G,i,n) values
c) Multiplying the (F/A,i,n) and (P/G,i,n) values
d) Multiplying the (P/G,i,n) and (A/F,i,n) values

(2-2) Yejen Industries Ltd. invested $10,000,000 in manufacturing equipment for producing small wastebaskets. If the company uses an interest rate of 15% per year, the amount of money it will have to earn each year to recover its investment in 7 years is closest to:

a) $2,403,600
b) $3,530,800
c) $3,941,800
d) $4,256,300

(2-3) The equivalent amount of money that can be spent seven years from now in lieu of spending $50,000 now at an interest rate 18% per year is closest to:

a) $15,700
b) $159,300
c) $199,300
d) $259,100

(2-4) Assume you borrow $10,000 today and promise to repay the loan in two payments, one in year 2 and the other in year 4, with the one in year 4 being only half as large as the one in year 2. At an interest rate of 10% per year, the size of the payment in year 4 will be closest to:

a) $4280
b) $3975
c) $3850
d) $2335

(2-5) You deposit $1000 now and you want the account to have a value as close to $8870 as possible in year 20. Assume the account earns interest at 10% per year. The year in which you must make another deposit of $1000 is

a) 6
b) 8
c) 10
d) 12

(2-6) The amount of money that Diamond Systems can spend now for improving productivity in lieu of spending $30,000 three years from now at an interest rate of 12% per year is closest to:

a) $15,700
b) $17,800
c) $19,300
d) $21,350

(2-7) A manufacturing company spent $30,000 on a new conveyor belt. If the conveyor belt resulted in cost savings of $4200 per year, the length of time it would take for the company to recover its investment at 8% per year is closest to:

a) Less than 9 years
b) 9 to 10 years
c) 11 to 12 years
d) Over 12 years

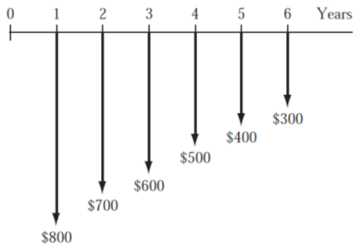
(2-8) Levi Strauss has some of its jeans stone-washed under a contract with independent U.S. Garment Corp. If U.S. Garment's operating cost per machine is $22,000 for year 1 and increases by a constant $1000 per year through year 5, what is the equivalent uniform annual cost per machine for the 5 years at an interest rate of 8% per year?

a) $23,850
b) $24,650
c) $25,930
d) Over $26,000

(2-9) The F/G factor values can be derived by multiplying:

a) (P/F) and (A/G) factor values
b) (F/P) and (A/G) factor values
c) (P/F) and (P/G) factor values
d) (F/P) and (P/G) factor values

(2-10) At i = 4% per year, A for years 1 through 6 of the cash flows shown below is closest to:



a) $300
b) $560
c) $800
d) $1040

(2-11) The value of the factor (P/F , i ,10) can be found by getting the factor values for (P/F , i, 4) and (P/F , i ,6) and:

a) Adding the values for (P/F, i, 4) and (P/F, i, 6)
b) Multiplying the values for (P/F, i, 4) and (P/F, i, 6)
c) Dividing the value for (P/F, i, 6) by the value for (P/F, i, 4)
d) None of the above

(2-12) A small construction company is considering the purchase of a used bulldozer for $61,000. If the company purchases the dozer now, the equivalent future amount in year 4 that the company is paying for the dozer at 4% per year interest is closest to:

a) $52,143
b) $65,461
c) $71,365
d) Over $72,000

(2-13) The cost of lighting and maintaining the tallest smokestack in the United States (at a shuttered ASARCO refinery) is $90,000 per year. At an interest rate of 10% per year, the present worth of maintaining the smokestack for 10 years is closest to:

a) $1,015,000
b) $894,000
c) $712,000
d) $553,000

(2-14) An enthusiastic new engineering graduate plans to start a consulting firm by borrowing $100,000 at 10% per year interest. The loan payment each year to pay off the loan in 7 years is closest to:

a) $18,745
b) $20,540
c) $22,960
d) $23,450

(2-15) An engineer who believed in "save now and play later" wanted to retire in 20 years with $1.5 million. At 10% per year interest, to reach the $1.5 million goal, starting 1 year from now, the engineer must annually invest:

a) $26,190
b) $28,190
c) $49,350
d) $89,680

(2-16) The cost of a border fence is $3 million per mile. If the life of such a fence is assumed to be 10 years, the equivalent annual cost of a 10-mile-long fence at an interest rate of 10% per year is closest to:

a) $3.6 million
b) $4.2 million
c) $4.9 million
d) Over $5.0 million

(2-17) An investment of $75,000 in equipment that will reduce the time for machining self-locking fasteners will save $20,000 per year. At an interest rate of 10% per year, the number of years required to recover the initial investment is closest to:

a) 6 years
b) 5 years
c) 4 years
d) 3 years

(2-18) The number of years required for an account to accumulate $650,000 if Ralph deposits $50,000 each year and the account earns interest at a rate of 6% per year is closest to:

a) 13 years
b) 12 years
c) 11 years
d) 10 years

(2-19) Aero Serve, Inc., manufactures cleaning nozzles for reverse-pulse jet dust collectors. The company spent $40,000 on a production control system that will increase profits by $13,400 per year for 5 years. The rate of return per year on the investment is closest to:

a) 20%
b) 18%
c) 16%
d) Less than 15%

(2-20) Energy costs for a green chemical treatment have been increasing uniformly for 5 years. If the cost in year 1 was $26,000 and it increased by $2000 per year through year 5, the present worth of the costs at an interest rate of 10% per year is closest to:

a) $102,900
b) $112,300
c) $122,100
d) $195,800

(2-21) In planning for your retirement, you expect to save $5000 in year 1, $6000 in year 2, and amounts increasing by $1000 each year through year 20. If your investments earn 10% per year, the amount you will have at the end of year 20 is closest to:

a) $242,568
b) $355,407
c) $597,975
d) $659,125

(2-22) Income from a precious metals mining operation has been decreasing uniformly for 5 years. If income in year 1 was $300,000 and it decreased by $30,000 per year through year 4, the annual worth of the income at 10% per year is closest to:

a) $310,500
b) $258,600
c) $203,900
d) $164,800

(2-23) If you are able to save $5000 in year 1, $5150 in year 2, and amounts increasing by 3% each year through year 20, the amount you will have at the end of year 20 at 10% per year interest is closest to:

a) $60,810
b) $102,250
c) $351,500
d) Over $410,000

Answers:

Question	***answer***
(2-1)	a
(2-2)	a
(2-3)	b
(2-4)	a
(2-5)	a
(2-6)	d
(2-7)	c
(2-8)	a
(2-9)	d
(2-10)	b
(2-11)	b
(2-12)	c
(2-13)	d
(2-14)	b
(2-15)	a
(2-16)	c
(2-17)	b
(2-18)	d
(2-19)	a
(2-20)	b
(2-21)	d
(2-22)	b
(2-23)	c

Chapter 3

Nominal and Effective Interest Rates

Chapter 3
Nominal and Effective Interest Rates

General Objective:

Trainee will be able to understand the Nominal and effective interest rates

Detailed Objectives:

1. Difference Between Nominal and Effective Interest Rates.
2. Calculating the Effective Interest Rate.
3. Formulation Equivalence Calculations Involving Only Single Amount Factors.
4. Equivalence Calculations Involving Series with PP ≥ CP and with PP < CP.

Introduction

In all engineering economy relations developed thus far, the interest rate has been a constant, annual value. For a substantial percentage of the projects evaluated by engineers in practice, the interest rate is compounded more frequently than once a year; frequencies such as semiannual, quarterly, and monthly are common. In fact, weekly, daily, and even continuous compounding may be experienced in some project evaluations. Also, in our own personal lives, many of the financial considerations we make—loans of all types (home mortgages, credit cards, automobiles, boats), checking and savings accounts, investments, stock option plans, etc.—have interest rates compounded for a time period shorter than 1 year. This requires the introduction of two new terms—nominal and effective interest rates. This chapter explains how to understand and use nominal and effective interest rates in engineering practice and in daily life situations.

1. Nominal and Effective Interest Rate Statements

In Chapter 1, we learned that the primary difference between simple interest and compound interest is that compound interest includes interest on the interest earned in the previous period, while simple interest does not. Here we discuss *nominal and effective interest rates,* which have the same basic relationship. The difference here is that the concepts of nominal and effective are used when interest is compounded more than once each year. For example, if an interest rate is expressed as 1% per month, the terms *nominal* and *effective* interest rates must be considered. Every nominal interest rate *must* be converted into an effective rate before it can be used in formulas, factor tables, calculator, or spreadsheet functions because they are all derived using effective rates.

The term *APR (Annual Percentage Rate)* is often stated as the annual interest rate for credit cards, loans, and house mortgages. This is the same as the *nominal rate*. An APR of 15% is the same as nominal 15% per year or a nominal 1.25% per month.

Also, the term *APY (Annual Percentage Yield)* is a commonly stated annual rate of return for investments, certificates of deposit, and savings accounts. This is the same as an *effective rate*. As we will discover, the nominal rate never exceeds the effective rate, and similarly APR < APY.

Before discussing the conversion from nominal to effective rates, it is important to *identify* a stated rate as either nominal or effective. There are three general ways of expressing interest rates as shown by the three groups of statements in Table (3-1). The three statements in the top third of the table show that an interest rate can be stated over some designated time period without specifying the compounding period. Such interest rates are assumed to be effective rates with the *compounding period (CP)* the same as that of the stated interest rate.

For the interest statements presented in the middle of Table (3-1), three conditions prevail: (1) The compounding period is identified, (2) this compounding period is shorter than the time period over which the interest is stated, and (3) the interest rate is designated neither as nominal nor as effective. In such cases, the interest rate is assumed to be *nominal* and the compounding period is equal to that which is stated. (We learn how to get effective interest rates from these in the next section).

For the third group of interest-rate statements in Table (3-1), the word *effective* precedes or follows the specified interest rate, and the compounding period is also given. These interest rates are obviously effective rates over the respective time periods stated.

The importance of being able to recognize whether a given interest rate is nominal or effective cannot be overstated with respect to the reader's understanding of the remainder of the material in this chapter and indeed the rest of the book. Table (3-2) contains a listing of several interest statements (column 1) along with their interpretations (columns 2 and 3).

Table (3-1): Various Interest Statements and Their Interpretations

(1) Interest Rate Statement	(2) Interpretation	(3) Comment
i = 12% per year i = 1% per month i = $3^{1/2}$% per quarter	i = *effective* 12% per year i = *effective* 1% per month compounded monthly i = *effective* $3^{1/2}$% per quarter compounded quarterly	When no compounding period is given, interest rate is an effective rate, with compounding period assumed to be equal to stated time period.
i = 8% per year compounded monthly i = 4% per quarter compounded monthly i = 14% per year compounded compounded semiannually	i = *nominal* 8% per year compounded monthly i = *nominal* 4% per quarter compounded monthly i = *nominal* 14% per year compounded compounded semiannually	When compounding period is given without stating whether the interest rate is nominal or effective, it is assumed to be nominal. Compounding period is as stated.
i = APY of 10% per year compounded monthly i = effective 6% per quarter i = effective 1% per month compounded daily	i = *effective* 10% per year compounded monthly i = *effective* 6% per quarter compounded quarterly i = *effective* 1% per month compounded daily	If interest rate is stated as an effective or APY rate, then it is an effective rate. If compounding period is not given, compounding period is assumed to coincide with stated time period.

Table (3-2): Specific Examples of Interest Statements and Interpretations

(1) Interest Rate Statement	(2) Nominal or Effective Interest	(3) Compounding Period
15% per year compounded monthly	Nominal	Monthly
15% per year	Effective	Yearly
Effective 15% per year compounded monthly	Effective	Monthly
20% per year compounded quarterly	Nominal	Quarterly
Nominal 2% per month compounded weekly	Nominal	Weekly
2% per month	Effective	Monthly
2% per month compounded monthly	Effective	Monthly
Effective 6% per quarter	Effective	Quarterly
Effective 2% per month compounded daily	Effective	Daily
1% per week compounded continuously	Nominal	Continuously

2. Effective Interest Rate Formulation

Understanding effective interest rates requires a definition of a nominal interest rate r as the interest rate per period times the number of periods. In equation form,

$$r = \ interest\ rate\ per\ period \times number\ of\ periods \qquad (3-1)$$

A nominal interest rate can be found for any time period that is longer than the compounding period. For example, an interest rate of 1.5% per month can be expressed as a *nominal* 4.5% per quarter (1.5% per period ×3 periods), 9% per semiannual period, 18% per year, or 36% per 2 years. Nominal interest rates obviously neglect compounding.

The equation for converting a nominal interest rate into an effective interest rate is

$$\boldsymbol{i\ per\ period = (1 + r/m)^{m} - 1 \qquad (3-2)}$$

where *i* is the *effective* interest rate for a certain period, say six months, *r* is the *nominal* interest rate for that *period* (six months here), and *m* is the number of times interest is *compounded in that same period* (six months in this case). The term *m* is often called the *compounding frequency*. As was true for nominal interest rates, effective interest rates can be calculated for any time period longer than the compounding period of a given interest rate. The next example illustrates the use of Equations (3-1) and (3-2).

Example (3-1)

1) A Visa credit card issued through Frost Bank carries an interest rate of 1% per month on the unpaid balance. Calculate the effective rate per semiannual and annual periods.
2) If the card's interest rate is stated as 3.5% per quarter, find the effective semiannual and annual rates.

Solution

1) The compounding period is monthly. For the effective interest rate per semiannual period, the r in Equation (3-1) must be the nominal rate per 6 months.

$$\begin{aligned} r &= 1\%\ per\ month \times 6\ months\ per\ semiannual\ period \\ &= 6\%\ \ per\ semiannual\ period \end{aligned}$$

The *m* in Equation (3-2) is equal to 6, since the frequency with which interest is compounded is 6 times in 6 months. The effective semiannual rate is

$$\begin{aligned} i\ per\ 6\ months &= (1 + 0.06/6)^{6} - 1 \\ &= 0.0615 = (6.15\%) \end{aligned}$$

For the effective annual rate, r = 12% per year and m = 12. By Equation (3-2),

$$\begin{aligned} Effective\ i\ per\ year &= (1 + 0.12/12)^{12} - 1 \\ &= 0.1268 = (12.68\%) \end{aligned}$$

2) For an interest rate of 3.5% per quarter, the compounding period is a quarter. In a semiannual period, m = 2 and r = 7%.

$$\begin{aligned} i\ per\ 6\ months &= (1 + 0.07/2)^{2} - 1 \\ &= 0.0712 = (7.12\%) \end{aligned}$$

The effective interest rate per year is determined using *r* = 14% and m = 4.

$$\begin{aligned} i\ per\ 6\ year &= (1 + 0.14/4)^{4} - 1 \\ &= 0.1475 = (14.75\%) \end{aligned}$$

If we allow compounding to occur more and more frequently, the compounding period becomes shorter and shorter. Then *m*, the number of compounding periods per payment period, increases. This situation occurs in businesses that have a very large number of cash flows every day, so it is correct to consider interest as compounded continuously. As *m* approaches infinity, the effective interest rate in Equation (3-2) reduces to

$$\boldsymbol{i = e^{r} - 1 \qquad (3-3)}$$

Equation (3-3) is used to compute the *effective continuous interest rate*. The time periods on *i* and *r* must be the same. As an illustration, if the nominal annual *r* = 15% *per year*, the effective continuous rate *per year* is

$$i\% = e^{0.15} - 1 = 16.183\%$$

For national and international chains—retailers, banks, etc.—and corporations that move thousands of items in and out of inventory each day, the flow of cash is essentially continuous. *Continuous cash flow* is a realistic model for the analyses performed by engineers and others in these organizations. The equivalence computations reduce to the use of integrals rather than summations. The topics of financial engineering analysis and continuous cash flows, coupled with continuous interest rates, are beyond the scope of this text; consult more advanced texts for formulas and procedures.

Example (3-2)

1) For an interest rate of 18% per year compounded continuously, calculate the effective monthly and annual interest rates.
2) An investor requires an effective return of at least 15%. What is the minimum annual nominal rate that is acceptable for continuous compounding?

Table (3-3): Effective Annual Interest Rates for Selected Nominal Rates

Nominal Rate r%	Semiannually (m = 2)	Quarterly (m = 4)	Monthly (m = 12)	Weekly (m = 252)	Daily (m = 365)	Continuously (m = ∞; e^r-1)
2	2.010	2.015	2.018	2.020	2.020	2.020
4	4 4.040	4.060	4.074	4.079	4.081	4.081
5	5.063	5.095	5.116	5.124	5.126	5.127
6	6.090	6.136	6.168	6.180	6.180	6.184
8	8.160	8.243	8.300	8.322	8.328	8.329
10	10.250	10.381	10.471	10.506	10.516	10.517
12	12.360	12.551	12.683	12.734	12.745	12.750
15	15.563	15.865	16.076	16.158	16.177	16.183
18	18.810	19.252	19.562	19.684	19.714	19.722

Solution

1) The nominal monthly rate is r = 18%/12 = 1.5%,or 0.015 per month. By Equation (3-3), the effective monthly rate is

$$i\%\ per\ month = e^{r} - 1 = e^{0.015} - 1 = 1.511\%$$

Similarly, the effective annual rate using r = 0.18 per year is

$$i\%\ per\ year = e^{r} - 1 = e^{0.18} - 1 = 19.72\%$$

2) Solve Equation (3-3) for r by taking the natural logarithm.

$$e^{r} - 1 = 0.15$$

$$e^{r} = 1.15$$

$$ln e^{r} = ln 1.15$$

$$r\% = 13.976\%$$

Therefore, a nominal rate of 13.976% per year compounded continuously will generate an effective 15% per year return.

Comment: The general formula to find the nominal rate, given the effective continuous rate i, is $r = \ln(1 + i)$

3. Reconciling Compounding Periods and Payment Periods

Now that the concepts of nominal and effective interest rates are introduced, in addition to considering the compounding period (which is also known as the interest period), it is necessary to consider the frequency of the payments or receipts within the cash-flow time interval. For simplicity, the frequency of the payments or receipts is known as the *payment period (PP)*. It is important to distinguish between the compounding period (CP) and the payment period because in many instances the two do not coincide. For example, if a company deposited money each month into an account that pays a nominal interest rate of 6% per year compounded semiannually, the payment period is 1 month while the compounding period is 6 months as shown in Figure (3-1). Similarly, if a person deposits money once each year into a savings account that compounds interest quarterly, the payment period is 1 year, while the compounding period is 3 months. Hereafter, for problems that involve either uniform-series or gradient cash-flow amounts, it will be necessary to determine the relationship between the compounding period and the payment period as a first step in the solution of the problem.

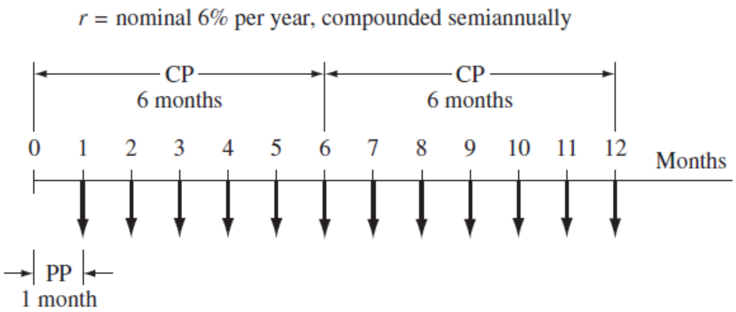


Figure (3-1): Cashflow diagram for a monthly payment period (PP) and semiannual compounding period (CP).

The next three sections describe procedures for determining the correct *i* and *n* values for use in formulas, and factor tables, as well as calculator and spreadsheet functions. In general, there are three steps:

1- Compare the lengths of PP and CP.
2- Identify the cash-flow series as involving only single amounts (*P* and *F*) or series amounts (*A*, *G*, or *g*).
3- Select the proper *i* and *n* values.

4. Formulation Equivalence Calculations Involving Only Single Amount Factors

There are many correct combinations of *i* and *n* that can be used when only singleamount factors (F/P and P/F) are involved. This is because there are only two requirements: (1) An effective rate must be used for *i*, and (2) the time unit on *n* must be the same as that on *i*. In standard factor notation, the single-payment equations can be generalized.

$$\boldsymbol{P = F(P/F, effective\ i\ per\ period, number\ of\ periods)} \qquad (3-4)$$

$$\boldsymbol{F = P(F/P, effective\ i\ per\ period, number\ of\ periods)} \qquad (3-5)$$

Thus, for a nominal interest rate of 12% per year compounded monthly, any of the i and corresponding n values shown in Table (3-4) could be used (as well as many others not shown) in the factors. For example, if an effective quarterly interest rate is used for i, that is, $(1.01)^3$-1 = 3.03%, then the n time unit is 4 quarters in a year.

Table (3-4): Various i and n Values for Single- Amount Equations Using r =12% per Year, Compounded Monthly

Effective Interest Rate, i	Units for n
1% per month	Months
3.03% per quarter	Quarters
6.15% per 6 months	Semiannual periods
12.68% per year	Years
26.97% per 2 years	2-year periods

Alternatively, it is always correct to determine the effective *i* per payment period using Equation (3-2) and to use standard factor equations to calculate *P*, *F*, or *A*.

Example (3-3)

Sherry expects to deposit $1000 now, $3000 4 years from now, and $1500 6 years from now and earn at a rate of 12% per year compounded semiannually through a company-sponsored savings plan. What amount can she withdraw 10 years from now?

Solution

Only single-amount P and F values are involved (Figure (3-2)). Since only effective rates can be present in the factors, use an effective rate of 6% per semiannual compounding period and semiannual payment periods. The future worth is calculated using Equation (3-5).

$$F = 1000(F/P, 6\%, 20) + 3000\ (F/P, 6\%, 12) + 1500\ (F/P, 6\%, 8) = \$11.634$$

An alternative solution strategy is to find the effective annual rate by Equation (3-2) and express *n* in years as determined from the problem statement.

$$Effective\ i\ per\ year = (1 + 0.12/2)^{2} - 1 = 0.1236\ (12.36\%)$$

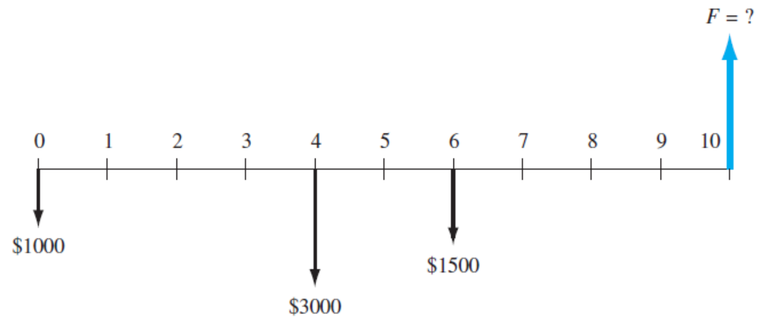


Figure (3-2): Cashflow diagram, Example (3-3)

5. Equivalence Calculations Involving Series with PP ≥ CP

When the cash flow of the problem dictates the use of one or more of the uniformseries or gradient factors, the relationship between the compounding period, CP, and payment period, PP, must be determined. The relationship will be one of the following three cases:

Type 1. Payment period equals compounding period, *PP = CP*.

Type 2. Payment period is longer than compounding period, *PP > CP*.

Type 3. Payment period is shorter than compounding period, *PP < CP*.

The procedure for the first two types is the same. Type 3 problems are discussed in the following section. When *PP = CP or PP > CP*, the following procedure *always* applies:

Step 1. Count the number of payments and use that number as *n*. For example, if payments are made quarterly for 5 years, *n* is 20.

Step 2. Find the *effective* interest rate over the *same time period* as *n* in step 1. For example, if *n* is expressed in quarters, then the effective interest rate per quarter *must* be used.

Use these values for *n* and *i* (and only these!) in the factors, functions, or formulas. To illustrate, Table (3-5) shows the correct standard notation for sample cash-flow sequences and interest rates. Note in column 4 that *n* is always equal to the number of payments and *i* is an effective rate expressed over the same time period as *n*.

Table (3-5): Examples of n and i Values Where PP = CP or PP > CP

(1) Cash-flow Sequence	(2) Interest Rate	(3) What to Find; What is Given	(4) Standard Notation
\$500 semiannually for 5 years	8% per year compounded semiannually	Find *P*; given *A*	*P*= 500(*P*/*A*,4%,10)
\$75 monthly for 3 years	12% per year compounded monthly	Find *F*; given *A*	*F* = 75 (*F*/*A*,1%,36)
\$180 quarterly for 15 years	5% per quarter	Find *F*; given *A*	*F* = 180 (*F*/*A*,5%,60)
\$25 per month increase for 4 years	1% per month	Find *P*; given *G*	*P* = 25 (*P*/*G*,1%,48)
\$5000 per quarter for 6 years	1% per month	Find *P*; given *A*	*P* = 5000 (*P*/*A*,3.03%,24)

Example (3-4)

For the past 7 years, a quality manager has paid \$500 every 6 months for the software maintenance contract on a laser-based measuring instrument. What is the equivalent amount after the last payment, if these funds are taken from a pool that has been returning 10% per year compounded quarterly?

Solution

The cash flow diagram is shown in Figure (3-3). The payment period (6 months) is longer than the compounding period (quarter); that is, PP > CP. Applying the guideline, determine an effective semiannual interest rate. Use Equation (3-2) or Table (3-3) with r = 0.05 per 6-month period and $m = 2$ quarters per semiannual period.

$$Effective\ i\ per\ 6\ months = (1-0.05/2)^{2}-1$$
$$= 5.063\%$$

The value i = 5.063% is reasonable, since the effective rate should be slightly higher than the nominal rate of 5% per 6-month period. The number of semiannual periods is n = 2(7) = 14 The future worth is

$$F = A(\mathrm{F/A}, 5.063\%, 14)$$
$$= 500\,(19.6845) = \$9842$$

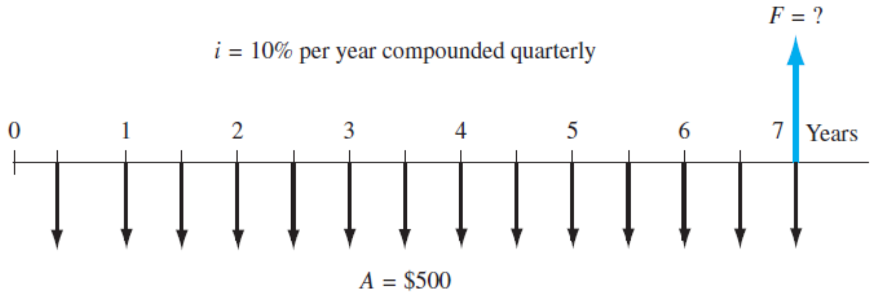


Figure (3-3): Diagram of semiannual payments used to determine F, Example (3-4)

6. Equivalence Calculations Involving Series with PP < CP

If a person deposits money each *month* into a savings account where interest is compounded *quarterly,* do the so-called *interperiod deposits* earn interest? The usual answer is no. However, if a monthly payment on a $10 million, quarterly compounded bank loan were made early by a large corporation, the corporate financial officer would likely insist that the bank reduce the amount of interest due, based on early payment. These two are examples of PP < CP, type 3 cash flows. The timing of cash flow transactions between compounding points introduces the question of how *interperiod compounding* is handled. Fundamentally, there are two policies: interperiod cash flows earn *no interest,* or they earn *compound interest*. The only condition considered here is the first one (no interest), because many realworld transactions fall into this category.

For a no-interperiod-interest policy, deposits (negative cash flows) are all regarded as *deposited at the end of the compounding period,* and withdrawals are all regarded as *withdrawn at the beginning.* As an illustration, when interest is compounded quarterly, all monthly deposits are moved to the end of the quarter, and all withdrawals are moved to the beginning (no interest is paid for the entire quarter). This procedure can significantly alter the distribution of cash flows before the effective quarterly rate is applied to find *P*, *F*, or *A*. This effectively forces the cash flows into a PP = CP situation, as discussed in Section 5.

Example (3-5)

Rob is the on-site coordinating engineer for Alcoa Aluminum, where an under- renovation mine has new ore refining equipment being installed by a local contractor. Rob developed the cash flow diagram in Figure (3-4a) in $1000 units from the project perspective. Included are payments to the contractor he has authorized for the current year and approved advances from Alcoa's home office. He knows that the interest rate on equipment "field projects" such as this is 12% per year compounded quarterly, and that Alcoa does not bother with interperiod compounding of interest. Will Rob's project finances be in the "red" or the "black" at the end of the year? By how much?

Solution

With no interperiod interest considered, Figure (3-4a) reflects the moved cash flows. The future worth after four quarters requires an *F* at an effective rate per quarter such that PP = CP = 1 quarter, therefore, the effective i = 12%/4 = 3%. Figure (3-4b) shows all negative cash flows (payments to contractor) moved to the end of the respective quarter, and all positive cash flows (receipts from home office) moved to the beginning of the respective quarter. Calculate the *F* value at 3%.

$$F = 1000[-150(F/P,3\%,4) - 200(F/P,3\%,3) + (-175 + 90)(F/P,3\%,2) + 165(F/P,3\%,1) - 50] = \$-357{,}592$$

Rob can conclude that the on-site project finances will be in the red about $357,600 by the end of the year.

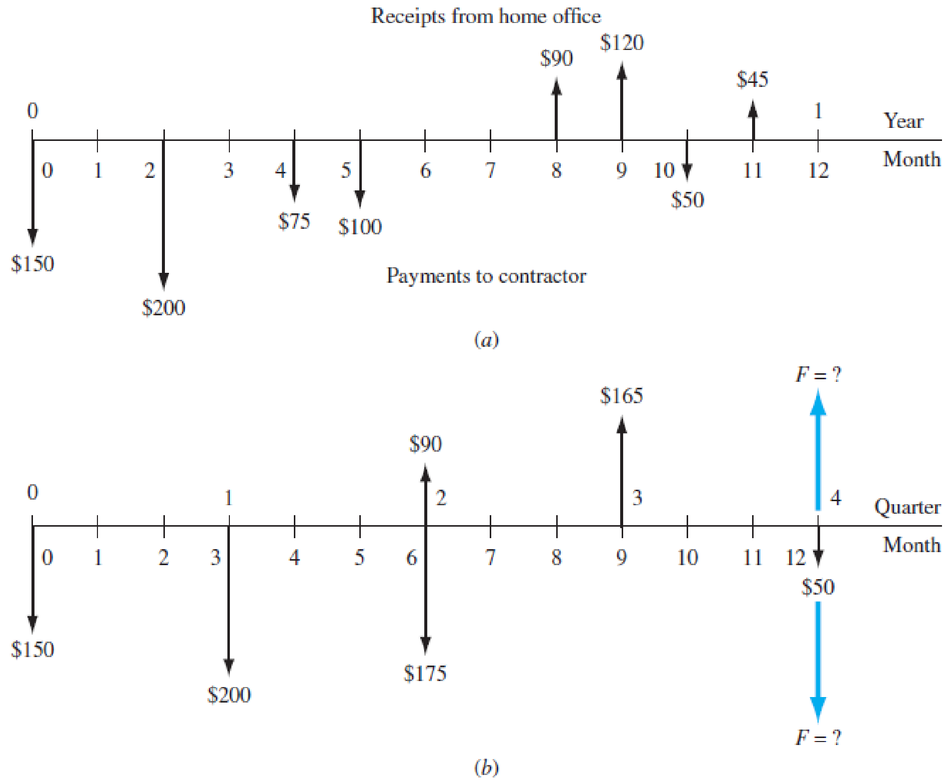


Figure (3-4): (a) Actual and (b) moved cash flows (in $1000) for quarterly compounding periods using no interperiod interest,, Example (3-5)

Problems

(3-1) For an interest rate of 2% per quarter, determine the nominal interest rate per:
a) Semiannual
b) Year
c) 2 years

(3-2) Identify each of the following interest rate statements as either nominal or effective.
a) 4% per year
b) 6% per year compounded annually
c) 10% per quarter
d) 8% per year compounded monthly
e) 1% per month
f) 1% per month compounded monthly
g) 0.1% per day compounded hourly
h) Effective 1.5% per month compounded weekly
i) 12% per year compounded semiannually
j) 1% per month compounded continuously

(3-3) A corporation deposits $20 million in a money market account for 1 year. What will be the difference in the total amount accumulated at the end of the year at 18% per year compounded monthly versus 18% per year simple interest?

(3-4) The TerraMax truck currently being manufactured and field tested by Oshkosh Truck Co. is a driverless truck intended for military use. Such a truck would free personnel for non-driving tasks such as reading maps, scanning for roadside bombs, or scouting for the enemy. If such trucks would result in reduced injuries to military personnel amounting to $15 million three years from now, determine the present worth of these benefits at an interest rate of 10% per year compounded semiannually.

(3-5) For the cash flows shown, determine the future worth in year 5 at an interest rate of 10% per year compounded continuously. Solve using :
a) Tabulated factors,
b) A financial calculator.

Year	1	2	3	4	5
Cash flow, $	300,000	0	250,000	0	200,000

(3-6) Improvements at a Harley-Davidson Plant are estimated to be $7.8 million. Construction is expected to take three years. What is the future worth of the project in year 3 at an interest rate of 6% per year compounded quarterly, assuming the funds are allocated
a) Completely at time 0,
b) Equally at the end of each year?

(3-7) Using tabulated factors for the cash flow series shown, calculate the future worth at the end of quarter 6 using i = 8% per year compounded quarterly. What are the calculator functions?

Quarter	0	1	2	3	4	5	6
Cash flow, $	100	100	300	300	300	300	300

(3-8) For the cash flows shown, determine the future worth in year 5 at an interest rate of 1% per month.

Year	1	2	3	4	5
Cash flow,	300,000	275,000	250,000	225,000	200,000

$

(3-9) In an effort to save money for early retirement, an environmental engineer plans to deposit $1200 per month starting one month from now, into a selfdirected investment account that pays 8% per year compounded semiannually. How much will be in the account at the end of 25 years?

(3-10) You plan to invest $1000 per month in a stock that pays a dividend at a rate of 4% per year compounded quarterly. How much will the stock be worth at the end of 9 years if there is no interperiod compounding?

(3-11) Western Refining purchased a model MTVS peristaltic pump for injecting antiscalant at its nanofiltration water conditioning plant. The cost of the pump was $1200. If the chemical cost is $11 per day, determine the equivalent cost per month at an interest rate of 1% per month. Assume 30 days per month and a 4-year pump life.

Chapter 4

Present Worth Analysis

Chapter 4
Present Worth Analysis

General Objective:

Trainee will be able to understand the Present Worth Analysis

Detailed Objectives:

1. Present Worth Analysis of Equal-Life Alternatives.
2. Present Worth Analysis of Different-Life Alternatives.
3. Capitalized Cost Analysis.
4. Evaluation of Independent Projects.
5. Using Spreadsheets for Present Worth Analysis.

Introduction

Alternatives are developed from project proposals to accomplish a stated purpose. The logic of alternative formulation and evaluation is depicted in Figure (4-1). Some projects are economically and technologically viable, and others are not. Once the viable projects are defined, it is possible to formulate the alternatives. Alternatives are one of two types: mutually exclusive or independent. Each type is evaluated differently.

- **Mutually exclusive (ME).** *Only one of the viable projects can be selected.* Each viable project *is* an alternative. If no alternative is economically justifiable, do nothing (DN) is the default selection.
- **Independent.** *More than one viable project may be selected* for investment. (There may be dependent projects requiring a particular project to be selected before another, and/or contingent projects where one project may be substituted for another.)

An alternative or project is comprised of estimates for the first cost, expected life, salvage value, and annual costs. Salvage value is the best estimate of an anticipated future market or trade-in value at the end of the expected life. Salvage may be estimated as a percentage of the first cost or an actual monetary amount; salvage is often estimated as nil or zero. Annual costs are commonly termed annual operating costs (AOC) or maintenance and operating (M&O) costs. They may be uniform over the entire life, increase or decrease each year as a percentage or arithmetic gradient series, or vary over time according to some other expected pattern.

A mutually exclusive alternative selection is the most common type in engineering practice. It takes place, for example, when an engineer must select the one best diesel-powered engine from several competing models. Mutually exclusive alternatives are, therefore, the same as the viable projects; each one is evaluated, and the one best alternative is chosen. Mutually exclusive alternatives *compete with one another* in the evaluation. All the analysis techniques compare mutually exclusive alternatives. Present worth is discussed in the remainder of this chapter.

The *do-nothing (DN)* option is usually understood to be an alternative when the evaluation is performed. If it is absolutely required that one of the defined alternatives be selected, do nothing is not considered an option. (This may occur when a mandated function must be installed for safety, legal, or other purposes.) Selection of the DN alternative means that the current approach is maintained; no new costs, revenues, or savings are generated.

Independent projects are usually designed to accomplish different purposes, thus the possibility of selecting any number of the projects. These projects (or bundles of projects) do not compete with one another; each project is evaluated separately, and the *comparison is* with the MARR.

Finally, it is important to classify an *alternative's* cash flows as revenue-based or cost-based. All alternatives evaluated in one study must be of the same type.

- **Revenue.** *Each alternative generates cost and revenue cash flow estimates, and possibly savings,* which are treated like revenues. Revenues may be different for each alternative. These alternatives usually involve new systems, products, and services that require capital investment to generate revenues and/or savings. Purchasing new equipment to increase productivity and sales is a revenue alternative.
- **Cost.** *Each alternative has only cost cash flow estimates.* Revenues are assumed to be equal for all alternatives. These may be public sector (government) initiatives, or legally mandated or safety improvements. Cost alternatives are compared to each

other; do-nothing is not an option when selecting from mutually exclusive cost alternatives.

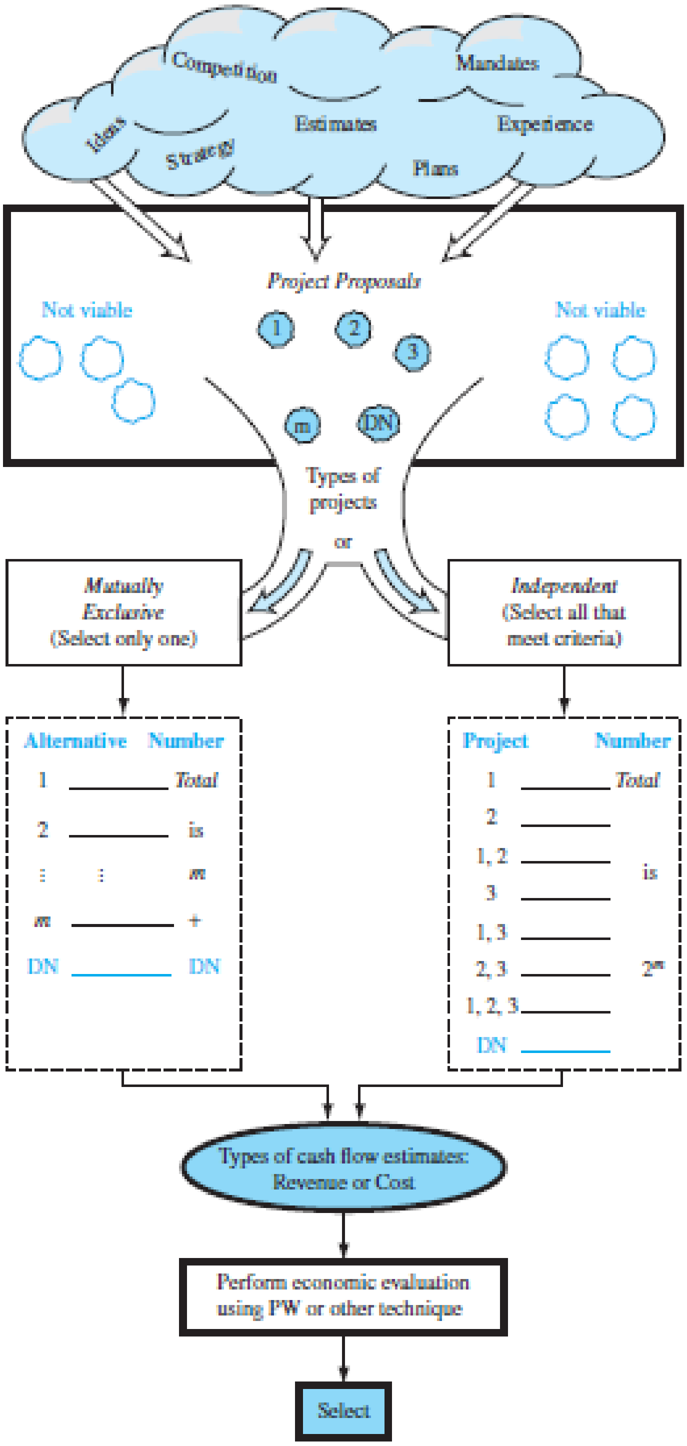


Figure (4-1): Logical progression from proposals to alternatives to selection.

1. Present Worth Analysis of Equal-Life Alternatives.

The PW comparison of alternatives with equal lives is straightforward. The present worth *P* is renamed PW of the alternative. The present worth method is quite popular in industry because all future costs and revenues are transformed to **equivalent monetary units NOW;**

that is, all future cash flows are converted (discounted) to present amounts (e.g., dollars) at a specific rate of return, which is the MARR. This makes it very simple to determine which alternative has the best economic advantage. The required conditions and evaluation procedure are as follows:

If the alternatives have the same capacities for the same time period (life), the equal-service requirement is met. Calculate the PW value at the stated MARR for each alternative.

For **mutually exclusive (ME)** alternatives, whether they are revenue or cost alternatives, the following guidelines are applied to justify a single project or to select one from several alternatives.

One alternative: If PW ≥ 0, the requested MARR is met or exceeded and the alternative is economically justified.

Two or more alternatives: Select the alternative with the PW that is numerically largest, that is, less negative or more positive. This indicates a lower PW of cost for cost alternatives or a larger PW of net cash flows for revenue alternatives.

Note that the guideline to select one alternative with the lowest cost or highest revenue uses the criterion of **numerically largest.** This is not the absolute value of the PW amount, because the sign matters. The selections below correctly apply the guideline for two alternatives A and B.

PW_A	PW_B	Selected Alternative
$-2300	$-1500	B
-500	+1000	B
+ 2500	+2000	A
+4800	-400	A

For **independent** projects, each PW is considered separately, that is, compared with the DN project, which always has PW = 0. The selection guideline is as follows:

One or more independent projects: Select all projects with PW ≥ 0 at the MARR.

The independent projects must have positive and negative cash fl ows to obtain a PW value that can exceed zero; that is, they must be revenue projects.

Example (4-1)

A university lab is a research contractor to NASA for in-space fuel cell systems that are hydrogenand methanol based. During lab research, three equal-service machines need to be evaluated economically. Perform the present worth analysis with the costs shown below. The MARR is 10% per year.

	Electric-Powered	Gas-Powered	Solar-Powered
First cost, $	-4500	-3500	-6000
Annual operating cost (AOC), $/year	-900	-700	-50
Salvage value *S*, $	200	350	100
Life, years	8	8	8

Solution

These are cost alternatives. The salvage values are considered a “negative” cost, so a + sign precedes them. (If it costs money to dispose of an asset, the estimated disposal cost has a -

sign.) The PW of each machine is calculated at i = 10% for n = 8 years. Use subscripts E , G , and S .

$$PW_E = -4500 - 900(P/A, 10\%, 8) + 200(P/F, 10\%, 8) = \$-9280$$

$$PW_G = -3500 - 700(\mathrm{P/A}, 10\%, 8) + 3500(\mathrm{P/F}, 10\%, 8) = \$-7071$$

$$PW_S = -6000 - 50(\mathrm{P/A}, 10\%, 8) + 100(\mathrm{P/F}, 10\%, 8) = \$-6220$$

The solar-powered machine is selected since the PW of its costs is the lowest; it has the numerically largest PW value.

Example (4-2)

As discussed in the introduction to this chapter, ultrapure water (UPW) is an expensive commodity for the semiconductor industry. With the options of seawater or groundwater sources, it is a good idea to determine if one system is more economical than the other. Use a MARR of 12% per year and the present worth method to select one of the systems.

Solution

An important first calculation is the cost of UPW per year. The general relation and estimated costs for the two options are as follows:

$$\boldsymbol{UPW\ cost\ relation}: = \frac{\$}{Year} = \left(\frac{cost\ in\ \$}{1000\ gallons}\right)\left(\frac{gallons}{minute}\right)\left(\frac{minutes}{hour}\right)\left(\frac{hours}{day}\right)\left(\frac{days}{year}\right)$$

$$Seawater: (4/1000)(1500)(60)(16)(250) = \$1.44\ M\ per\ year$$

$$Groundwater: (5/1000)(1500)(60)(16)(250) = \$1.80\ M\ per\ year$$

Calculate the PW at i = 12% per year and select the option with the lower cost (larger PW value). In \$1 million units:

$$\boldsymbol{PW\ relation}: PW = first\ cost - PW\ of\ AOC - PW\ of\ UPW + PW\ of\ salvage\ value$$

$$PW_S = -20 - 0.5(P/A, 12\%, 10) - 1.44(P/A, 12\%, 10) + 0.05(20)(P/F, 12\%, 10)$$
$$= -20 - 0.5(5.6502) - 1.44(5.6502) + 1(0.3220) = \$-30.64$$

$$PW_G = -22 - 0.3(P/A, 12\%, 10) - 1.80(P/A, 12\%, 10) + 0.10(22)(P/F, 12\%, 10)$$
$$= -22 - 0.2(5.6502) - 1.80(5.6502) + 2.2(0.3220) = \$-33.16$$

Based on this present worth analysis, the seawater option is cheaper by \$2.52 M.

2. Present Worth Analysis of Different-Life Alternatives

When the present worth method is used to compare mutually exclusive alternatives that have different lives, the equal-service requirement must be met.

The PW of the alternatives must be compared over the same number of years and must end at the same time to satisfy the equal-service requirement.

This is necessary, since the present worth comparison involves calculating the equivalent PW of all future cash flows for each alternative. A fair comparison requires that PW values represent cash flows associated with equal service. For cost alternatives, failure to compare equal service will always favor the shorter-lived mutually exclusive alternative, even if it is not the more economical choice, because fewer periods of costs are involved. The equal-service requirement is satisfied by using either of two approaches:

LCM: *Compare the PW of alternatives over a period of time equal to the* ***least common multiple (LCM)*** *of their estimated lives.*

Study period: *Compare the PW of alternatives using a* ***specified study period of n years.***

This approach does not necessarily consider the useful life of an alternative. The study period is also called the planning horizon.

For either approach, calculate the PW at the MARR and use the same selection guideline as that for equal-life alternatives. The LCM approach makes the cash flow estimates extend to the same period, as required. For example, lives of 3 and 4 years are compared over a 12-year period. The first cost of an alternative is reinvested at the beginning of each life cycle, and the estimated salvage value is accounted for at the end of each life cycle when calculating the PW values over the LCM period. Additionally, the LCM approach requires that some assumptions be made about subsequent life cycles.

The assumptions when using the LCM approach are that

1. ***The service provided will be needed over the entire LCM years or more.***
2. ***The selected alternative can be repeated over each life cycle of the LCM in exactly the same manner.***
3. ***Cash flow estimates are the same for each life cycle.***

A study period analysis is necessary if the first assumption about the length of time the alternatives are needed cannot be made. For the study period approach, a time horizon is chosen over which the economic analysis is conducted, and only those cash flows which occur during that time period are considered relevant to the analysis. All cash flows occurring beyond the study period are ignored. An estimated market value at the end of the study period must be made. The time horizon chosen might be relatively short, especially when short-term business goals are very important. The study period approach is often used in replacement analysis. It is also useful when the LCM of alternatives yields an unrealistic evaluation period, for example, 5 and 9 years.

Example (4-3)

National Homebuilders, Inc., plans to purchase new cut-and-fi nish equipment. Two manufacturers offered the estimates below.

	Vendor A	Vendor B
First cost, $	-15,000	-18,000
Annual M&O cost (AOC), $/year	-3,500	-3,100
Salvage value, $	1,000	2,000
Life, years	6	9

1- Determine which vendor should be selected on the basis of a present worth comparison, if the MARR is 15% per year.

2- National Homebuilders has a standard practice of evaluating all options over a 5-year period. If a study period of 5 years is used and the salvage values are not expected to change, which vendor should be selected?

Solution

1- Since the equipment has different lives, compare them over the LCM of 18 years. For life cycles after the first, the first cost is repeated in year 0 of each new cycle, which is the last year of the previous cycle. These are years 6 and 12 for vendor A and year 9 for B. The cash flow diagram is shown in Figure (4–2). Calculate PW at 15% over 18 years.

$$PW_A = -15.000 - 15.000(P/F, 15\%, 6) + 1000(P/F, 15\%, 6)$$
$$-15.000(P/F, 15\%, 12) + 1000(P/F, 15\%, 12) + 1000(P/F, 15\%, 18)$$
$$-3.500(P/A, 15\%, 18) = \$ - 45.036$$

$$PW_B = -18.000 - 18.000(P/F, 15\%, 9) + 2000(P/F, 15\%, 9)$$
$$+2000(P/F, 15\%, 18) - 3100(P/A, 15\%, 8) = \$ - 41.384$$

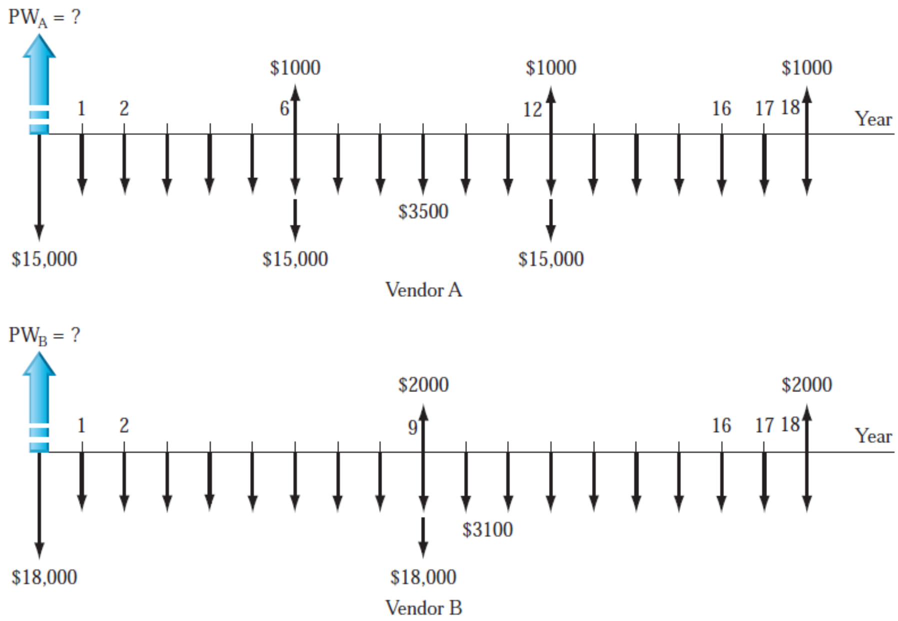


Figure (4-2): Cash flow diagram for different-life alternatives, Example (4-3).

Vendor B is selected, since it costs less in PW terms; that is, the PW_B value is numerically larger than PW_A.

2- For a 5-year study period, no cycle repeats are necessary. The PW analysis is

Fdhd

$$PW_A = -15.000 - 3500(P/A, 15\%, 5) + 1000(P/F, 15\%, 5) = \$ - 26.236$$
$$PW_B = -18.000 - 3100(P/A, 15\%, 5) + 2000(P/F, 15\%, 5) = \$ - 27.397$$

Vendor A is now selected based on its smaller PW value. This means that the shortened study period of 5 years has caused a switch in the economic decision. In situations such as this, the standard practice of using a fixed study period should be carefully examined to ensure that the appropriate approach, that is, LCM or fixed study period, is used to satisfy the equal-service requirement.

3. Capitalized Cost Analysis

Many public sector projects such as bridges, dams, highways and toll roads, railroads, and hydroelectric and other power generation facilities have very long expected useful lives. A **perpetual or infinite life** is the effective planning horizon. Permanent endowments for charitable organizations and universities also have perpetual lives. The economic worth of these types of projects or endowments is evaluated using the present worth of the cash flows.

> ***Capitalized Cost (CC)*** *is the present worth of a project that has a very long life (more than, say, 35 or 40 years) or when the planning horizon is considered very long or infinite.*

The formula to calculate CC is derived from the PW relation P/A (P/A , i %, n), where $n = \infty$ time periods. Take the equation for P using the P/A factor and divide the numerator and denominator by $(1 + i)^n$ to obtain

$$P = A\left[\frac{1-\frac{1}{(1+i)^n}}{1}\right]$$

As n approaches ∞, the bracketed term becomes $1/i$. We replace the symbols P and PW with CC as a reminder that this is a capitalized cost equivalence. Since the A value can also be termed AW for annual worth, the capitalized cost formula is simply

$$CC = \frac{A}{i} or\ CC = \frac{AW}{i} \quad (4-1)$$

Solving for A or AW, the amount of new money that is generated each year by a capitalization of an amount CC is

$$AW = CC(i) \quad (4-2)$$

This is the same as the calculation *A/P(i)* for an infinite number of time periods. Equation (5-2) can be explained by considering the time value of money. If $20,000 is invested now (this is the capitalization) at 10% per year, the maximum amount of money that can be withdrawn at the end of every year for eternity is $2000, which is the interest accumulated each year. This leaves the original $20,000 to earn interest so that another $2000 will be accumulated the next year. The cash flows (costs, revenues, and savings) in a capitalized cost calculation are usually of two types: recurring, also called periodic, and nonrecurring. An annual operating cost of $50,000 and a rework cost estimated at $40,000 every 12 years are examples of recurring cash flows. Examples of nonrecurring cash flows are the initial investment amount in year 0 and one-time cash flow estimates at future times, for example, $500,000 in fees 2 years hence.

The procedure to determine the CC for an infinite sequence of cash flows is as follows:

1. Draw a cash flow diagram showing all nonrecurring (one-time) cash flows and at least two cycles of all recurring (periodic) cash flows.
2. Find the present worth of all nonrecurring amounts. This is their CC value.
3. Find the A value through *one life cycle* of all recurring amounts. (This is the same value in all succeeding life cycles) Add this to all other uniform amounts (A) occurring in years 1 through infinity. The result is the total equivalent uniform annual worth (AW).

4. Divide the AW obtained in step 3 by the interest rate *i* to obtain a CC value. This is an application of Equation (4-1).
5. Add the CC values obtained in steps 2 and 4.

Drawing the cash flow diagram (step 1) is more important in CC calculations than elsewhere, because it helps separate nonrecurring and recurring amounts. In step 5 the present worths of all component cash flows have been obtained; the total capitalized cost is simply their sum.

Example (4-4)

The Haverty County Transportation Authority (HCTA) has just installed new software to charge and track toll fees. The director wants to know the total equivalent cost of all future costs incurred to purchase the software system. If the new system will be used for the indefinite future, find the equivalent cost:

a- now, a CC value,
b- for each year hereafter, an AW value.

The system has an installed cost of $150,000 and an additional cost of $50,000 after 10 years. The annual software maintenance contract cost is $5000 for the first 4 years and $8000 thereafter. In addition, there is expected to be a recurring major upgrade cost of $15,000 every 13 years. Assume that i = 5% per year for county funds.

Solution

a- The five-step procedure to fi nd CC now is applied.

1. Draw a cash flow diagram for two cycles Figure (4–3).
2. Find the present worth of the nonrecurring costs of $150,000 now and $50,000 in year 10 at i = 5%. Label this CC_1

$$CC_1 = -150{,}000 - 50{,}000(P/F, 5\%, 10) = \$-180{,}695$$

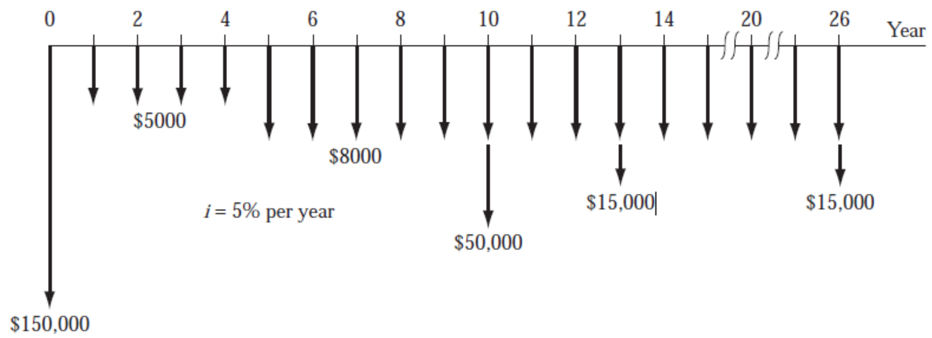


Figure (4-3): Cash flows for two cycles of recurring costs and all nonrecurring amounts, Example (4-4).

3. **And 4** .Convert the $15,000 recurring cost to an A value over the first cycle of 13 years, and find the capitalized cost CC_2 at 5% per year using Equation (4-1).

$$A = -15{,}000(A/F, 5\%, 13) = \$-847$$

$$CC_2 = -780/0.05 = \$-16{,}940$$

There are several ways to convert the annual software maintenance cost series to *A* and CC values. A straightforward method is to, first, consider the $−5000 an *A* series with a capitalized cost of

$$CC_3 = -5000/0.05 = \$-100{,}000$$

Second, convert the step-up maintenance cost series of $−3000 to a capitalized cost CC_4 in year 4, and find the present worth in year 0. (Refer to Figure (4–4) for cash flow timings.)

$$CC_4 = (-3000/0.05)(P/F, 5\%, 4) = \$-49{,}362$$

5. The total capitalized cost CC_T for Haverty County Transportation Authority is the sum of the four component CC values.

$$CC_T = -180{,}695 - 16{,}940 - 100{,}000 - 49{,}362 = \$-346{,}997$$

b- Equation (4-2) determines the AW value forever.

$$AW = P_i = CC_T(i) = -346{,}997(0.05) = \$17{,}350$$

Correctly interpreted, this means Haverty County officials have committed the equivalent of $17,350 forever to operate and maintain the toll management software.

4. Evaluation of Independent Projects.

Consider a biomedical company that has a new genetics engineering product that it can market in three different countries (S, U, and R), including any combination of the three. The do-nothing (DN) alternative is also an option. All possible options are: S, U, R, SU, SR, UR, SUR, and DN. In general, for *m* independent projects, there are 2^m alternatives to evaluate. Selection from independent projects uses a fundamentally different approach from that for mutually exclusive (ME) alternatives. When selecting independent projects, each project's PW is calculated using the MARR. (In ME alternative evaluation, the projects compete with each other, and only one is selected.) The selection rule is quite simple for one or more *independent* projects:

Select all projects that have PW ≥ 0 at the MARR.

All projects must be developed to have revenue cash flows (not costs only) so that projects earning more than the MARR have positive PW values.
Unlike ME alternative evaluation, which assumes the need for the service over multiple life cycles, independent projects are considered one-time investments. This means the PW analysis is performed over the respective life of each project and the assumption is made that any leftover cash flows earn at the MARR when the project ends. As a result, the equal service requirement does not impose the use of a specified study period or the LCM method. The implied study period is that of the longest lived project.
There are two types of selection environments—unlimited and budget constrained.

- **Unlimited.** All projects that make or exceed the MARR are selected. Selection is made using the PW ≥ 0 guideline.
- **Budget constrained.** No more than a specified amount, *b*, of funds can be invested in all of the selected projects, and each project must make or exceed the MARR. Now the solution methodology is slightly more complex in that *bundles* of projects that do not exceed the investment limit *b* are the only ones evaluated using PW values. The procedure is:
  1. Determine all bundles that have total initial investments no more than *b*. (This limit usually applies in year 0 to get the project started).
  2. Find the PW value at the MARR for all projects contained in the bundles.
  3. Total the PW values for each bundle in (1).
  4. Select the bundle with the largest PW value.

5. Using Spreadsheets for Present Worth Analysis.

Spreadsheet- or calculator-based evaluation of equal-life, mutually exclusive alternatives can be performed using the single-cell PV function when the annual amount *A* is the same. The general format to determine the PW is

$$= P - PV(i, n, A, F) \quad (4-3)$$

It is important to pay attention to the sign placed on the PV function in order to get the correct answer for the alternative's PW value. The spreadsheet function returns the opposite sign of the *A* series. Therefore, to retain the negative sense of a cost series *A*, place a minus sign immediately in front of the PV function. This is illustrated in the next example.

Example (4-5)

Cesar, a petroleum engineer, has identified two equivalent diesel-powered generators to be purchased for an offshore platform. Use *i* = 12% per year to determine which is the more economic. Solve using both spreadsheet and calculator functions.

	Generator 1	Generator 2
P, $	-80,000	-120,000
S, $	15,000	40,000
n, years	3	3
AOC, $/year	-30,000	-8,000

Solution

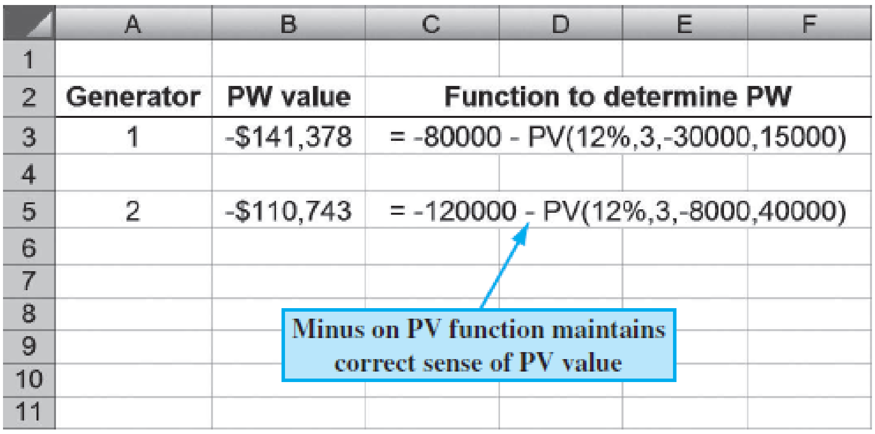
Spreadsheet: Follow the format in Equation (4-3) in a single cell for each alternative. Figure (4-4) shows the details. Note the use of minus signs on *P*, the PV function, and AOC value. Generator 2 is selected with the smaller PW of costs (numerically larger value).

Calculator: The function and PW value for each alternative are:

Generator 1: -80000 - PV(12,3,-30000,15000) PW1 = $-141,378

Generator 2: -120000 - PV(12,3,-8000,40000) PW2 = $-110,743

As expected, the PW values and selection of Generator 2 are the same as the spreadsheet solution.



	A	B	C	D	E	F
1						
2	Generator	PW value	Function to determine PW			
3	1	-$141,378	= -80000 - PV(12%,3,-30000,15000)			
4						
5	2	-$110,743	= -120000 - PV(12%,3,-8000,40000)			
6						
7						
8						
9						
10						
11						

Figure (4-4): Equal-life alternatives evaluated using the PV function, Example (4-5).

When different-life alternatives are evaluated, using the LCM basis, it is necessary to input all the cash flows for the LCM of the lives to ensure an equal-service evaluation. Develop the NPV function to find PW. If cash flow is identified by CF, the general format is

$$= P + NPV(i\%, year\_1\_CF\_cell: last\_year\_CF\_cell) \quad (4-4)$$

It is very important that the *initial cost P not be included* in the cash flow series identified in the NPV function. Unlike the PV function, the NPV function returns the correct sign for the PW value.

Example (4-6)

Continuing with the previous example, once Cesar had selected generator 2 to purchase, he approached the manufacturer with the concerns that the first cost was too high and the expected life was too short. He was offered a lease arrangement for 6 years with a $20,000 annual cost and an extra $20,000 payment in the first and last years to cover installation and removal costs. Determine if generator 2 or the lease arrangement is better at 12% per year.

	A	B	C	D	E	F	G
1	Year	Generator 2	Lease				
2	0	-120,000	-40,000				
3	1	-8,000	-20,000				
4	2	-8,000	-20,000				
5	3	-88,000	-20,000				
6	4	-8,000	-20,000				
7	5	-8,000	-20,000				
8	6	32,000	-40,000				
9	PW value	-$189,568	-$132,361				
10							
11							
12							

Repurchase cash flow
= S - AOC - P
= 40,000 – 8,000 – 120,000

= –40000 + NPV(12%,C3:C8)

= –120000 + NPV(12%,B3:B8)

Figure (4-5):Different-life alternatives evaluated using the NPV function, Example (4-6).

Solution

Assuming that generator 2 can be repurchased 3 years hence and all estimates remain the same, PW evaluation over 6 years is correct. Figure 4.5 details the cash flows and NPV functions. The year 3 cash flow for generator 2 is *S* - AOC - *P* = $ - 88,000. Note that the first costs are not included in the NPV function but are listed separately, as indicated in Equation (4-4). The lease option is the clear winner for the next 6 years.

When evaluating alternatives for which the annual cash flows do not form an *A* series, the individual amounts must be entered on the spreadsheet and Equation (4-4) is used to find PW values. Also, remember that any zero-cash-flow year must be entered as 0 to ensure that the NPV function correctly tracks the years.

Problems

(4-1) State two conditions under which the do-nothing alternative is not an option.

(4-2) What is the difference between mutually exclusive alternatives and independent projects?

(4-3) When evaluating projects by the present worth method, how do you know which one(s) to select, if the

a) Projects are independent,
b) Alternatives are mutually exclusive?

(4-4) A biomedical engineer with Johnston Implants just received estimates for replacement equipment to deliver online selected diagnostic results to doctors performing surgery who need immediate information on the patient's condition. The cost is $200,000, the annual maintenance contract costs $5000, and the useful life (technologically) is 5 years.

a) What is the alternative if this equipment is not selected? What other information is necessary to perform an economic evaluation of the two?
b) What type of cash flow series will these estimates form?
c) What additional information is needed to convert the cash flow estimates to the other type of series?

(4-5) A company that manufactures magnetic membrane switches is investigating two production options that have the estimated cash fl ows shown ($1 million units). Which one should be selected on the basis of a present worth analysis at 10% per year?

	In-house	Contract
First cost, $	-30	0
Annual cost, $ per year	-5	-2
Annual income, $ per year	14	3.1
Salvage value, $	2	--
Life, years	5	5

(4-6) What is meant by the term equal service?

(4-7) Oil from a particular type of marine microalgae can be converted to biodiesel that can serve as an alternate transportable fuel for automobiles and trucks. If lined ponds are used to grow the algae, the construction cost will be $13 million and the maintenance & operating (M&O) cost will be $2.1 million per year. If long plastic tubes are used for growing the algae, the initial cost will be higher at $18 million, but less contamination will render the M&O cost lower at $0.41 million per year. At an interest rate of 10% per year and a 5-year project period, which system is better, ponds or tubes? Use a present worth analysis.

(4-8) A wealthy businessman wants to start a permanent fund for supporting research directed toward sustainability. The donor plans to give equal amounts of money for each of the next 5 years, plus one now (i.e., six donations) so that $100,000 per year can be withdrawn each year forever, beginning in year 6. If the fund earns interest at a rate of 8% per year, how much money must be donated each time?

(4-9) The cost of maintaining a certain permanent monument in Washington, DC occurs as periodic outlays of $1000 every year and $5000 every 4 years. Calculate the capitalized cost of the maintenance using an interest rate of 10% per year.

Compare the alternatives shown on the basis of their capitalized costs using an interest rate of 10% per year.

	Alternative M	Alternative N
First cost, $	-150,000	-800,000
Annual cost, $ per year	-50,000	-12,000
Annual income, $ per year		
Salvage value, $	8,000	1,000,000
Life, years	5	∞